ELECTRICAL POWER TRANSMISSION

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PRINCIPLES OF DESIGN AND PERFORMANCE

BY

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PREFACE

The material of this volume represents the outgrowth of a course in Transmission Line Theory and Design given to senior and graduate students in electrical engineering at the University of Washington during the past ten years. Most of the material comprising the last half of the volume was developed and carefully tested in the field of practical line design. Throughout the book the emphasis is put upon the development, so far as possible, of a complete rational theory. Details of a practical nature, which can be better had from handbooks, from the manufacturer of items of equipment, or may be had only by experience, are for the most part omitted.

The first seven chapters are devoted largely to a discussion of the mathematical tools and the underlying circuit theory of the transmission line, including dielectric, magnetic, and electrical circuits One brief chapter only is devoted to approximate circuits and their calculation, since adequate discussions of these are usually available in books on alternating-current theory. Throughout the book the emphasis is put upon the exact rather than approximate methods. There are a number of good reasons for this. This method is the only correct one; it is not particularly complex; it is easily applied where calculating machines are available, as is now quite generally the case; it serves as a basis for further study of both communication and power-circuit problems; and, finally, it offers a splendid opportunity to apply a little advanced circuit theory, with respect to which no self-respecting advanced student in electrical engineering can afford to remain in ignorance.

The last seven chapters are devoted to a discussion of the mechanical features of line design, a study of the economics of line design as influenced by both the mechanical and electrical features, and, finally, to the working out of an illustrative example in which the principles and theory previously developed are applied to the complete solution of a hypothetical design problem.

The mechanical features of transmission lines are often touched upon very lightly or are entirely neglected in books on electric

power transmission. Yet it is recognized that the design of the span and the proper selection of towers are important features of the problem of transmission line design. While very little detailed discussion is given pertaining to the types of supports available or their proper design from the standpoint of strength, yet three chapters are devoted to the discussion of the theory and application of span design and their bearing upon the selection of the most economical line. A method is outlined and illustrated by means of which the most economical type of structure may be chosen.

The matter of economics as affecting engineering problems is a subject which is rarely mentioned to engineering students, and yet the practicing engineer fully realizes that in nearly every engineering problem it is a prime consideration. This phase of the problem of line design is strongly featured. Kelvin's law of maximum economy, modified to fit the conditions more completely, is made the basis upon which the size of conductor and the line voltage are chosen. By the use of empirical equations developed from appropriate engineering data, a mathematical statement of Kelvin's modified law is evolved, from which reliable design results are obtained. The method has been tested in a number of practical line designs with entire success.

The writer has drawn material and inspiration from many sources. Where the specific source is apparent it is usually acknowledged in a footnote. Much of our knowledge of a given subject, however, represents a gradual accumulation over a long period of time; its subject matter is the contribution of many workers in the field; its origin is obscure or unknown. Such material has been used without specific mention.

Much of the work of the last seven chapters is based on articles which originally appeared either in the *Journal* or the *Transactions* of the A.I.E.E., in bulletins of the University of Washington Engineering Experiment Station, or in the *Electrical World*. These papers, written for the most part by the author and his associates, Professor F. K. Kirsten and Assistant Professor G. S. Smith, are acknowledged in footnotes appearing in the text.

When this volume was originally conceived, it was planned to bring it out under the joint authorship of Professor Kirsten and the writer. This accounts in part for the fact that Chap. XII, particularly, follows Professor Kirsten's work very closely. The writer regrets that the press of other work made it impossible for Professor Kirsten to participate as originally planned.

In acknowledging his indebtedness to the colleagues above mentioned, the author desires also to express his gratitude for their interest in the progress of his work, for permission to use their material, and for their helpful suggestions and criticisms The author is likewise indebted to Mr. Roy E. Lindblom for carefully reading and checking the manuscript.

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CHAPTER I

COMPLEX QUANTITIES, CIRCULAR AND HYPERBOLIC FUNCTIONS

The Mathematics of Transmission-line Design.—In the electrical calculation of long transmission-line circuits as well as in the preparation of temperature-tension stringing charts required for the proper stringing of the line conductors, a rigorous solution requires the use of certain mathematical relations which are not usually well understood by the average engineer. This lack of understanding arises, perhaps, partly because much of the engineer's everyday work does not usually require mathematics beyond a fair working knowledge of algebra and trigonometry, and partly from the fact that these more powerful mathematical tools have come into more or less general use only during recent years, and are therefore unfamiliar to many engineers who received their college training before that time.

In special cases, as in the calculation of the performance curves for short lines or the stringing charts for short spans, approximate solutions involving better known and somewhat simpler mathematical relations may give results to a degree of accuracy well within the limits of practical requirements. For long lines and long spans, however, the approximate solutions will not give reliable results, and more accurate methods are essential. After all, the only final test of the accuracy of any approximate method is a method which is rigorously correct. For the sake of completeness it seems desirable to include in this chapter a brief discussion of complex numbers, even though these are usually better understood than some of the other mathematical relations included in transmission line problems.

Complex Quantities.—Algebraic numbers such as 2, 4, x, $-m$ are one-dimensional quantities and may be considered as locating

a point in a line with reference to a fixed starting point or origin (Fig. 1). Thus $+4$, $+7$, $+x$ locate points of corresponding numbers of units to the right of zero, while $-m$ indicates a point m units to the left of zero. Ordinary numbers of this type, however, will locate points on a line such as OX only. These numbers are called *real numbers,* and the X-axis is chosen as the axis of reals.

In a similar manner points may be located in the direction taken at right angles to OX by the use of ordinary numbers measured in the Y-direction. Numbers measured in the Y-

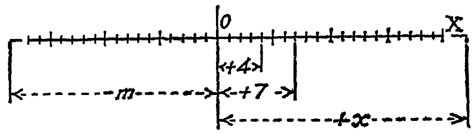

FIG 1.—Real numbers locate points on a line.

direction are called *quadrature numbers* or *imaginaries,* and are written with the operator j prefixed to the magnitude, as $j4$, $-j5$, $+jy$, etc. Quadrature numbers are also one-dimensional and serve to locate points above or below the X-axis. Thus the Y-axis becomes the quadrature or imaginary axis.

By combining the real and the quadrature quantities into a single quantity as $a + jb$, $4 + j5$, $x + jy$, a *complex quantity* is obtained. A complex quantity is thus a two-dimensional quantity which serves to locate a point anywhere within a given plane. For example, the quantity

$$OP = 3 + j4 \tag{1}$$

locates the point P (Fig. 2) in the plane XY three units to the right of the line OY and four units above the line OX. The point P is then completely located in the plane XY by giving its two coordinates, namely, ($x = 3$) and ($y = 4$). Likewise, the vector quantity OP is determined by giving its x and y components ($ox = 3$ and $oy = 4$, respectively), and Eq. (1) may be interpreted to mean that the vector quantity OP is made up of a $+x$ component $= 3$, and a $+y$ component $= 4$. Thus the length of the vector, as well as its position in the plane, is fixed.

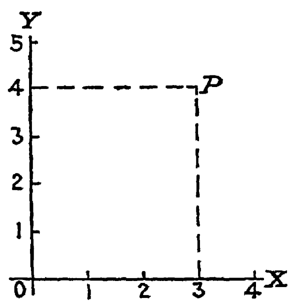

FIG. 2.—Complex numbers locate points in a plane.

The Operator j.—The symbol j is a symbol of operation and, like other symbols of operation, denotes that a certain operation is to be performed on the quantity with which it is associated. Common symbols of operation are log, $\sqrt{\ }$, $\int$, as well as numbers written in the form $2 \cdot 3 \cdot 5 \cdot 1$. Each of these symbols commands us to do a certain thing to the quantity following the symbol. Thus $\sqrt{2}$ is interpreted to mean an operation which, when performed twice, is equivalent to doubling, that is, an operation such that $\sqrt{2} \times \sqrt{2} = 2$. The numbers $2 \cdot 3 \cdot 5 \cdot 1$ indicate that 1 is to be taken five times, the resulting 5 is to be taken three times, making 15, and the resulting 15 taken twice, making 30. Thus each of the numbers becomes a symbol of operation. The numerical operators 2, 3, 5, etc., serve to stretch or increase the magnitude of quantities with which they are associated and are sometimes called *tensors*. The operator -1, on the other hand, does not affect the magnitude of its associated quantity but simply reverses the sense of the magnitude, and this operator is therefore a *reversor*. Consistent with this idea 1×4 means four units to the right of the origin along the axis of reals, and -1×4 means four units to the left of the origin on the same axis. A reversal without change in magnitude may be conceived of as a rotation of a fixed magnitude through 180°, the counter-clockwise direction being taken as positive for convenience. Since such rotation is accomplished by the reversor -1, one may think of the operator $\sqrt{-1}$ as an operator which, when applied twice, will produce a rotation of 180° since $\sqrt{-1} \times \sqrt{-1} = -1$. Thus the expression $\sqrt{-1}$, for which the symbol j is used in engineering literature, may be defined as an operator which turns a fixed magnitude through 90° in a positive or counter-clockwise direction. This symbol is used to denote quadrature quantities, as already explained.

Powers of j—Powers of j higher than the first, which arise from performing the ordinary algebraic operations on vector or complex quantities, may be easily eliminated. This is readily understood from a consideration of the significance attached to the operator. A single application of the operator produces a positive rotation of 90°; two applications produce a rotation of $2 \times 90°$, or 180°, equivalent to a reversal; three applications produce a rotation of 270°, which is equivalent to a negative rotation of 90°; and four applications of the operator produce a complete revolution of the vector, equivalent to a rotation

of zero degrees. Repeated applications of $-j$ produce corresponding rotations in the reverse or negative direction. Accordingly, the higher powers of j carry the following meanings:

$$\begin{aligned}
+j &= \quad\quad\quad\; + 90^\circ \text{ rotation} = +j \\
+j^2 &= \quad\quad\quad\; +180^\circ \text{ rotation} = -1 \\
+j^3 &= \quad\quad\quad\; +270^\circ \text{ rotation} = -j \\
+j^4 &= \quad\quad\quad\; +360^\circ \text{ rotation} = +1 \\
+j^5 &= +360^\circ + 90^\circ \text{ rotation} = +j, \text{ etc.}
\end{aligned}$$

Also,

$$+j \times -j = -j^2 = 0^\circ \text{ rotation} = +1.$$

Rectangular and Polar Forms.—Complex quantities are very conveniently used to represent vectors such as forces, velocities, accelerations, voltages, currents, etc. Maximum and effective values of sinusoidal alternating currents and voltages of the same frequency are conveniently represented by such quantities, and their use in electrical calculations has become quite general To illustrate, suppose the currents $\dot{I}_1$ and $\dot{I}_2$ of a branched circuit combine to form the current $\dot{I}_3$, and let

$$\dot{I}_1 = 20 + j30 \tag{2}$$
$$\dot{I}_2 = 40 - j10 \tag{3}$$

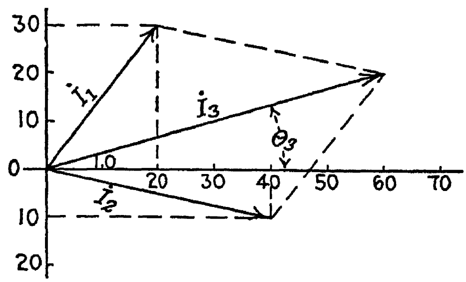

FIG. 3.—Addition of complex or vector numbers.

These currents are graphically represented in Fig. 3. The sum of the two currents is

$$\begin{aligned}
\dot{I}_3 &= \dot{I}_1 + \dot{I}_2 \\
&= (20 + 40) + j(30 - 10) \\
&= 60 + j20
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
\theta_3 &= \tan^{-1} \frac{20}{60} \\
&= \underline{/18^\circ\, 26'}.
\end{aligned}$$

The vector current $\dot{I}_3$ is thus completely defined by giving the magnitudes of its real and quadrature components and the angle of inclination of the resulting vector as related to a chosen reference axis, usually the X-axis. The form of expression used in Eq. (4) is most commonly used to represent vector quantities in rectangular coordinates. Since the length of $\dot{I}_3$ is

$$\begin{aligned} I_3 &= \sqrt{(60)^2 + (20)^2} \\ &= 63.2 \text{ amp} \end{aligned}$$

the complex quantity for $\dot{I}_3$ may also be written

$$\dot{I}_3 = I_3 (\cos 18° \, 26' + j \sin 18° \, 26') \qquad (5)$$
$$= 63.2 \, \underline{/18° \, 26'}. \qquad (6)$$

(Equations (5) and (6) are polar expressions for the complex quantity $\dot{I}_3$)

The notation used above is that ordinarily followed; that is, the vector quantity is written with a dotted capital as

$$\dot{I}_3 = I_1 + jI_2$$

or

$$\dot{E} = E_1 + jE_2, \text{ etc.,}$$

while the magnitude or length of the vector is indicated by the capital letter without the dot, thus,

$$\begin{aligned} I_3 &= \sqrt{I_1^2 + I_2^2} \\ E &= \sqrt{E_1^2 + E_2^2}, \text{ etc.} \end{aligned}$$

This method of differentiating between a vector and its length is helpful for beginners, but is an unnecessary refinement for the more advanced reader, since the form of the equation or other information available in the context will clearly indicate whether the vector quantity or its length is meant. Therefore, in the later chapters of this book the dot will not be used.

Equation (5) gives the current $\dot{I}_3$ in terms of its polar components, and this form is regarded as the polar expression for the complex quantity. The symbol $\underline{/\quad\quad}$ is used by some writers to indicate the quadrant in which the angle lies. Thus we have the following symbols:

First quadrant $\underline{/\quad\quad}$
Second quadrant $\underline{\backslash\quad\quad}$
Third quadrant $\overline{/\quad\quad}$
Fourth quadrant $\overline{\backslash\quad\quad}$

Frequently, however, only the symbol for the first quadrant is used, with the angle inserted, as, for example, $\underline{/-\delta}$, $\underline{/26° 40'}$, $\underline{/-10° 30'}$, etc.

Addition and Subtraction.—The advantage of using complex quantities in rectangular coordinates in the solution of problems involving vectors is, perhaps, most evident when the addition or subtraction of two or more vectors is required. In the illustration (Fig. 3) the resultant current $\dot{I}_3$ is found by adding separately the real and the quadrature components of $\dot{I}_1$ and $\dot{I}_2$.

Or, in general, if

$$\dot{I}_1 = a_1 + jb_1$$
$$\dot{I}_2 = a_2 + jb_2.$$

Adding,

$$\dot{I}_3 = (a_1 + a_2) + j(b_1 + b_2) \tag{7}$$

or

$$I_3 = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$$

and

$$\theta_3 = \tan^{-1} \frac{b_1 + b_2}{a_1 + a_2}.$$

The corresponding polar form is

$$\dot{I}_3 = I_3 \underline{/\tan^{-1} \frac{b_1 + b_2}{a_1 + a_2}}.$$

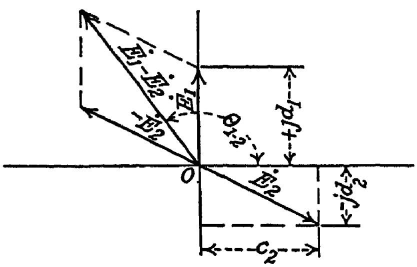

FIG. 4 —Subtraction of complex or vector numbers.

Subtraction of vectors is accomplished by reversing the sign of one of the vectors and adding as before. In Fig. 4 let the voltages, $\dot{E}_1$ and $\dot{E}_2$ be two of the three leg voltages of a three-phase circuit. Then

$$\dot{E}_1 = 0 + jd_1$$
$$\dot{E}_2 = c_2 - jd_2.$$

Subtracting,

$$\dot{E}_{1-2} = \dot{E}_1 - \dot{E}_2 \tag{8}$$
$$= -c_2 + j(d_1 + d_2) \tag{9}$$
$$E_{1-2} = \sqrt{c^2 + (d_1 + d_2)^2}$$
$$\theta_{1-2} = \tan^{-1} \frac{d_1 + d_2}{-c_2}$$

or

$$\dot{E}_{1-2} = E_{1-2}\underline{\left|\tan^{-1}\frac{d_1 + d_2}{-c_2}\right.} \text{ vector volts.} \qquad (10)$$

$\dot{E}_{1-2}$ is the potential difference between the lines 1 and 2. The length of this vector is E_{1-2} and its angle of inclination referred to the X-axis is θ_{1-2}.

Multiplication.—Let it be required to multiply the two vectors A and B of Fig. 5, where

$$\dot{A} = 8 + j4 \qquad A = 8.94 \qquad \theta_A = 26°\,34'$$
$$\dot{B} = 4 + j3 \qquad B = 5.0 \qquad \theta_B = 36°\,52'.$$

Multiplying,

$$\dot{C} = \dot{A}\dot{B} = (32 + j16 + j24 + j^2 12) \qquad (11)$$

and since

$$+j^2 = -1$$
$$\dot{C} = 20 + j40$$
$$C = \sqrt{20^2 + 40^2}$$
$$= \sqrt{2000} = 44.7$$

and

$$\dot{C} = 44.7\underline{\left/\tan^{-1}\frac{40}{20}\right.}$$
$$= 44.7\underline{/63°\,26'}. \qquad (12)$$

The magnitude of the resultant vector $\dot{C}$ thus obtained is the same as that obtained by taking the product of the magnitudes of A and B.

For

$$C = A \times B$$
$$= 5 \times 8.94$$
$$= 44.7.$$

It will also be observed that

$$\theta_C = \theta_a + \theta_b$$
$$= 26°\,34' + 36°\,52'$$
$$= 63°\,26'.$$

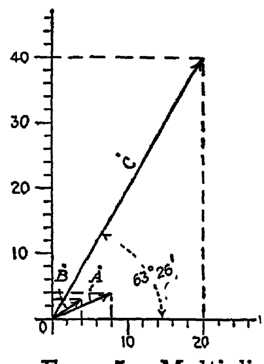

FIG. 5.—Multiplication of complex or vector numbers.

In the above is presented an illustration of the law of multiplication of complex quantities. This law may be stated as follows: *The product of two complex quantities is a complex quantity whose*

length is equal to the product of the lengths of the original quantities, and whose angle of inclination is the sum of the original angles.

The proof of this law is readily established by the use of the polar notation.

For, if

$$\dot{A} = A\,(\cos\theta_1 + j\sin\theta_1) \tag{13}$$

and

$$\dot{B} = B(\cos\theta_2 + j\sin\theta_2) \tag{14}$$

then

$$\begin{aligned}\dot{C} = \dot{A}\dot{B} &= AB(\cos\theta_1\cos\theta_2 + j\sin\theta_1\cos\theta_2 \\ &\qquad + j\cos\theta_1\sin\theta_2 + j^2\sin\theta_1\sin\theta_2) \\ &= AB\,(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 \\ &\qquad + \cos\theta_1\sin\theta_2) \\ &= AB\,[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)].\end{aligned} \tag{15}$$

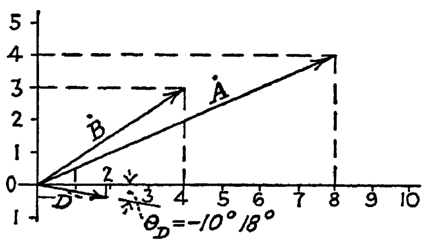

FIG 6.—Division of complex or vector numbers.

Division.—If it be required to divide $\dot{A}$ by $\dot{B}$ the division (Fig. 6) is stated thus,

$$\dot{D} = \frac{\dot{A}}{\dot{B}} = \frac{8 + j4}{4 + j3}.$$

In order to carry out the operation here indicated it is necessary to rationalize the denominator. This is done by multiplying both numerator and denominator of the fraction by the conjugate[1] of the denominator.

Carrying out this operation,

$$\dot{D} = \frac{8 + j4}{4 + j3} \times \frac{4 - j3}{4 - j3} \tag{16}$$

$$= \frac{32 + j16 - j24 - j^2 12}{16 + j12 - j12 - j^2 9}$$

and since $-j^2 = +1$

$$\dot{D} = \frac{44 - j8}{25}$$

$$= 1.76 - j0.32 \tag{17}$$

[1] Two complex quantities are said to be conjugate if they differ only in the sign of the quadrature term. For example, the complex numbers $a + jb$ and $a - jb$ are conjugate.

$$D = \sqrt{1.76^2 + 0.32^2}$$
$$= 1.79 \diagdown \tan^{-1}\frac{0.32}{1\ 76}$$
$$= 1.79 \diagdown \overline{10^\circ\ 18'}. \tag{18}$$

The length of the vector resulting from the division is length A $\div$ length B;
for

$$\frac{A}{B} = \frac{8.94}{5}$$
$$= 1.79$$

and its angle of inclination is

$$\theta_D = \theta_A - \theta_B$$
$$= (26^\circ\ 34') - (36^\circ\ 52')$$
$$= \diagdown \overline{10^\circ\ 18'} \text{ or } \diagdown \underline{-(10^\circ\ 18')}.$$

The law for the division of complex quantities here illustrated is: *The division of two complex quantities yields a complex quantity whose magnitude is the quotient derived by dividing the magnitudes of the original quantities, and whose angle of inclination is the angle of the dividend less the angle of the divisor.*

The proof of this law is similar to that for the law of multiplication. Writing the polar equations in the general form;

$$\dot{A} = A\ (\cos\theta_1 + j\sin\theta_1) \tag{19}$$
$$\dot{B} = B\ (\cos\theta_2 + j\sin\theta_2) \tag{20}$$

$$\dot{D} = \frac{\dot{A}}{\dot{B}}$$
$$= \frac{A}{B}\left[\frac{\cos\theta_1 + j\sin\theta_1}{\cos\theta_2 + j\sin\theta_2}\right]$$
$$= \frac{A}{B}\left[\frac{\cos\theta_1 + j\sin\theta_1}{\cos\theta_2 + j\sin\theta_2} \times \frac{\cos\theta_2 - j\sin\theta_2}{\cos\theta_2 - j\sin\theta_2}\right]$$
$$=$$
$$\frac{A}{B}\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) - j(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right]$$
$$= \frac{A}{B}[\cos(\theta_1 - \theta_2) - j\sin(\theta_1 - \theta_2)]$$
$$= \frac{A}{B}\underline{/\theta_1 - \theta_2}. \tag{21}$$

Powers.—The nth power of a complex quantity is found by taking the quantity n times as a factor. Raising a given complex quantity to any power is in reality a special case of multiplication.

Let it be required to find the nth power of the quantity

$$\dot{A} = A\ (\cos\theta_1 + j\sin\theta_1) \tag{22}$$

Then

$$\dot{A}^n = A^n\ (\cos\theta_1 + j\sin\theta_1)^n \tag{23}$$

and, by the law for the multiplication of complex numbers,

$$\dot{A}^n = A^n \cos(\theta_1 + \theta_1 + \theta_1 + \cdot \qquad) + j\sin(\theta_1 + \theta_1 + \theta_1 + \ \cdot\ \cdot\ \cdot\ ) \tag{24}$$

to n terms
or

$$\begin{aligned}\dot{A}^n &= A^n\ (\cos n\theta_1 + j\sin n\theta_1) \\ &= A^n \underline{/n\theta_1}\end{aligned} \tag{25}$$

since

$$\theta_n = n\theta_1 \tag{26}$$

from which it follows that

$$(\cos\theta_1 + j\sin\theta_1)^n = (\cos n\theta_1 + j\sin n\theta_1). \tag{27}$$

This relation is known as De Moivre's theorem.

Thus, *when raising a complex quantity to any power n, the result is a complex quantity whose magnitude is the nth power of the original quantity, and whose angle of inclination is n times the original angle.* To illustrate, let it be required to find the third power of the complex number $5 + j2$. In polar form (Fig. 7),

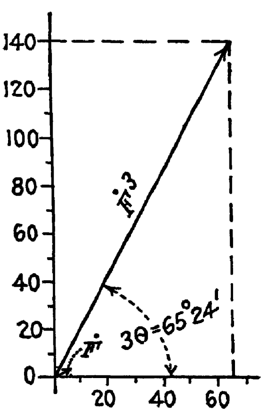

Fig. 7.—Powers of complex or vector numbers.

$$\dot{F} = (5 + j2) = 5.38\ (\cos 21°\ 48' + j\sin 21°\ 48') \tag{28}$$

and

$$\begin{aligned}\dot{F}^3 &= (5 + j2)^3 \\ &= 5.38^3[\cos 3(21°\ 48') + j\sin 3\ (21°\ 48')] \\ &= 5.38^3\ (\cos 65°\ 24' + j\sin 65°\ 24') \\ &= 155.7(0.416 + j0.909) \\ &= 64.8 + j141.5 \\ &= 155.6\underline{/65°\ 24'}.\end{aligned} \tag{29}$$

Roots.—It may be shown that De Moivre's theorem holds for both integral and fractional values of n. Using fractional values of n, this theorem furnishes a simple tool for finding the roots of complex numbers.

If

$$\dot{A} = A(\cos\theta_1 + j\sin\theta_1)$$

then the nth root of this expression is

$$\sqrt[n]{\dot{A}} = \mathrm{A}^{\frac{1}{n}}(\cos\theta_1 + j\sin\theta_1)^{\frac{1}{n}} \tag{30}$$

$$= A^{\frac{1}{n}}\left((\cos\frac{\theta_1}{n} + j\sin\frac{\theta_1}{n}\right). \tag{31}$$

Thus, *the nth root of a complex quantity is a complex quantity whose magnitude is the nth root of the original quantity, and whose angle of inclination is $\frac{1}{n}$th of the original angle.*

For example, let it be required to evaluate the expression $\sqrt{4 + j3}$.

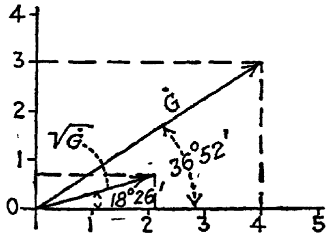

FIG. 8.—Roots of complex or vector numbers.

Let

$$\dot{G} = 4 + j3$$

$$= 5\underline{/\tan^{-1}\frac{3}{4}}$$

$$= 5\underline{/36° \; 52'})$$

Then

$$\sqrt{\dot{G}} = \sqrt{5}\,\underline{/\tfrac{1}{2}(36° \; 52')}$$

$$= 2.23\underline{/18° \; 26'}$$

$$= 2.12 + j0.705 \text{ (Fig. 8).}$$

Circular Functions.—In rectangular coordinates, the equation of an ellipse whose center is at the origin of the coordinate axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{32}$$

where a and b represent the major and minor half axes. If, in this equation, a is made equal to b, there results the special case of the ellipse in which the two axes are equal, that is, the circle. Writing $a = b = r$, the equation of the circle is

$$x^2 + y^2 = r^2 \tag{33}$$

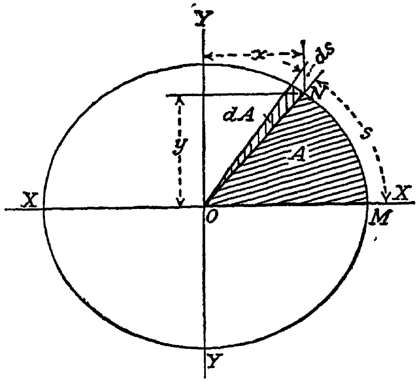

FIG 9.—The circular angle θ

The radius r is of constant length, and, as it sweeps around from the position OM (Fig. 9), the point N traces out arc S of the circle, while the radius itself sweeps out the area A of a sector. The length ds of the arc S is proportional to the area of the sector dA, and the angle $d\theta$ through which the radius moves may therefore be measured either in terms of the length of arc or in terms of the area of the sector, *i.e.*,

$$\text{length of arc} = ds = rd\theta$$

or

$$d\theta = \frac{ds}{r}$$

and

$$\theta = \frac{1}{r}\int_0^S ds = \frac{s}{r}. \qquad (34)$$

If the radius be chosen as the unit of measure, the arc and the angle become numerically equal, or

$$d\theta = ds$$

and

$$\theta = \int_0^S ds$$
$$= S.$$

Also, the area of the sector is

$$dA = \frac{rds}{2}$$
$$= \frac{r^2 d\theta}{2}$$

and

$$d\theta = \frac{2dA}{r^2} \qquad (35)$$

or, for $r = 1$,

$$d\theta_1 = 2dA$$
$$\theta_1 = 2\int_0^A dA \qquad (36)$$
$$= 2A.$$

Thus a circular angle θ is measured either by its arc or by twice the area of the corresponding sector.

The relation

$$\theta = \frac{\text{length of arc}}{\text{length of radius}} = \frac{S}{r} \tag{37}$$

is called the circular measure of an angle and, for this reason, trigonometric functions of an angle are often called circular functions. The ratio may be looked upon as a percentage. The angle then becomes the percentage of the length of the arc in terms of the constant-length radius taken as unity. The principal circular functions are

$$\left.\begin{aligned} \frac{y}{r} &= \sin\frac{S}{r} = \sin\theta \\ \frac{x}{r} &= \cos\frac{S}{r} = \cos\theta \\ \frac{y}{x} &= \tan\frac{S}{r} = \tan\theta \end{aligned}\right\} \tag{38}$$

Hyperbolic Functions.—In rectangular coordinates the equation of an hyperbola whose center is at the origin of the coordinate axes is

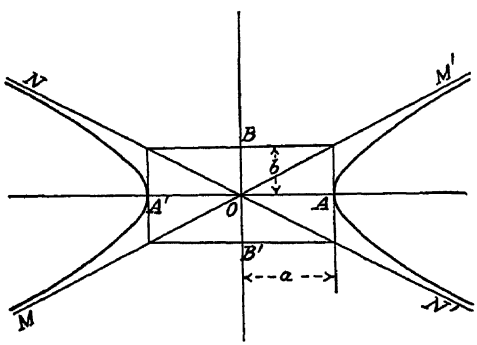

FIG. 10—The hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{39}$$

where a and b are equal to OA and OB of Fig. 10, respectively.

The lines MM' and NN' are asymptotes to the two branches of the hyperbola. If, in Eq. (39), a be put equal to b, there results the rectangular hyperbola whose equation is

$$x^2 - y^2 = a^2. \tag{40}$$

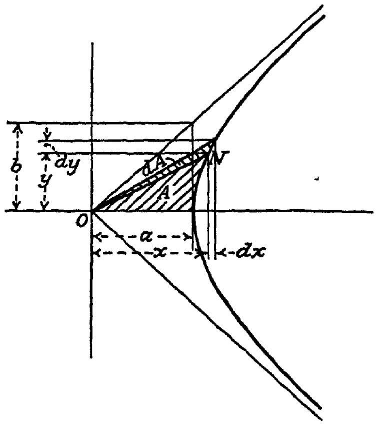

FIG. 11.—The hyperbolic angle u.

One branch of this hyperbola is pictured in Fig. 11.

If ds represents a small length of arc such that when x changes by an amount dx, y changes a corresponding amount dy and s by an equivalent amount ds, then

$$(ds)^2 = (dx)^2 + (dy)^2$$

or

$$ds = dx\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$$

If ρ is the radius vector ON, then writing

$$\frac{ds}{\rho} = du,$$

$$du = \frac{dx}{\rho}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \tag{41}$$

and

$$u = \int_a^x \frac{dx}{\rho}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}. \tag{42}$$

If a be taken as the unit of measure, Eq. (42) becomes

$$u = \int_1^x \frac{dx}{\rho}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}. \tag{43}$$

The function u is the integrated ratio of the length of an infinitesimal arc to the length of the radius vector ρ at any point P. Therefore, since

$$u = \int \frac{\text{length of arc}}{\text{length of radius}}$$

by analogy to the circular angle θ, the function u is called an *hyperbolic angle.* It is to be noted that, while in the circle r is

constant, here the radius vector ρ is a variable. Also, analogous to the circle, it may be shown that the function u is measured by twice the area of the sector.

$$\rho = \sqrt{x^2 + y^2}$$

and from Eq. (40),

$$y^2 = x^2 - a^2. \tag{44}$$

Differentiating Eq. (44),

$$2x \cdot dx = 2y \cdot dy$$

or

$$\frac{dy}{dx} = \frac{x}{y}. \tag{45}$$

Substituting these values in Eq. (42),

$$\begin{aligned} u &= \int_a^x \frac{dx}{\sqrt{x^2 - a^2}} \\ &= \log_\epsilon[x + \sqrt{x^2 - a^2}] - \log_\epsilon a \\ &= \log_\epsilon \frac{x + \sqrt{x^2 - a^2}}{a} \end{aligned} \tag{46}$$

or

$$a\epsilon^u = x + \sqrt{x^2 - a^2}. \tag{47}$$

Solving,

$$x = \frac{a(\epsilon^u + \epsilon^{-u})}{2}$$

and

$$\frac{x}{a} = \frac{\epsilon^u + \epsilon^{-u}}{2}. \tag{48}$$

Since

$$y^2 = x^2 - a^2$$

by substituting the value of x from Eq. (48), it follows that

$$\begin{aligned} y^2 &= \frac{a^2}{4}(\epsilon^{2u} + 2\epsilon^u\epsilon^{-u} + \epsilon^{-2u} - 4) \\ &= \frac{a^2}{4}(\epsilon^u - \epsilon^{-u})^2 \end{aligned}$$

and

$$\frac{y}{a} = \frac{\epsilon^u - \epsilon^{-u}}{2}. \tag{49}$$

By dividing Eq. (49) by Eq. (48), there results

$$\frac{y}{x} = \frac{\epsilon^u - \epsilon^{-u}}{\epsilon^u + \epsilon^{-u}}. \tag{50}$$

In the case of the circle, the ratios $\frac{x}{a}$, $\frac{y}{a}$ and $\frac{y}{x}$ are called the cosine, sine, and tangent of θ, respectively. For the rectangular hyperbola these ratios are called respectively the cosine, sine, and tangent of the hyperbolic angle u. In abbreviated form they are written,

$$\sinh u = \frac{y}{a} = \frac{\epsilon^u - \epsilon^{-u}}{2} \tag{51}$$

$$\cosh u = \frac{x}{a} = \frac{\epsilon^u + \epsilon^{-u}}{2} \tag{52}$$

$$\tanh u = \frac{y}{a} = \frac{\epsilon^u - \epsilon^{-u}}{\epsilon^u + \epsilon^{-u}} \tag{53}$$

Sine, Cosine and Exponential Series.—By Maclaurin's theorem a function of a single variable may be expanded into a series of ascending powers of the variable. If y is a function of x, only, then, by Maclaurin's theorem,

$$y = f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \quad . \quad . \tag{54}$$

where A, B, C, D, etc., are constants whose values are independent of x but dependent upon the constants contained in the original expression of $f(x)$. In order to make the expression on the right-hand side of Eq. (54) definite, the values of the constants must be determined. This is done by successive differentiation.

By successive differentiation of Eq. (54) the following equations are obtained:

$$\frac{dy}{dx} = \frac{d}{dx}\cdot f(x) = 0 + B + 2Cx + 3Dx^2 + 4Ex^3 + \quad . \quad . \tag{55}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\cdot f'(x) = 0 + 0 + 1.2C + 2.3Dx + 3.4Ex^2 + \quad . \quad . \tag{56}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\cdot f''(x) = 0+0+0+1\cdot 2\cdot 3\cdot D+2\cdot 3\cdot 4\cdot Ex+ \quad . \quad . \quad . \tag{57}$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\cdot f'''(x) = 0 + 0 + 0 + 0 + 1\cdot 2\cdot 3\cdot 4E + \quad . \quad . \quad . \tag{58}$$

Since these equations hold for all values of x, they hold for $x = 0$, whence, substituting $x = 0$ in each equation and solving,

$$
\begin{array}{llll}
y = f(0) = A & & A = y \\
\dfrac{dy}{dx} = f'(0) = 1 \cdot B & & B = \dfrac{dy}{dx} \\
\dfrac{d^2y}{dx^2} = f''(0) = 1 \cdot 2C & & C = \dfrac{1}{1 \cdot 2} \cdot \dfrac{d^2y}{dx^2} \\
\dfrac{d^3y}{dx^3} = f'''(0) = 1 \cdot 2 \cdot 3 \cdot D & & D = \dfrac{1}{1 \cdot 2 \cdot 3} \cdot \dfrac{d^3y}{dx^3} \\
\dfrac{d^4y}{dx^4} = f''''(0) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot E & & E = \dfrac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \dfrac{d^4y}{dx^4}
\end{array}
$$

Applying this theorem to the expansion of $\cos x$,

$$y = \cos x = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ . \ . \tag{59}$$

and

$$
\begin{aligned}
A &= y = \cos 0 = 1 \\
B &= \frac{dy}{dx} = -\sin 0 = 0 \\
C &= \frac{d^2y}{dx^2} = \frac{-\cos 0}{\underline{|2}} = \frac{-1}{\underline{|2}} \\
D &= \frac{d^3y}{dx^3} = \frac{+\sin 0}{\underline{|3}} = 0 \\
E &= \frac{d^4y}{dx^4} = \frac{+\cos 0}{\underline{|4}} = \frac{1}{\underline{|4}}, \text{ etc.}
\end{aligned}
$$

Substituting these values in Eq. (59),

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \frac{x^8}{\underline{|8}} - \ . \ . \ . \tag{60}$$

By similar methods, it is found that

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \frac{x^9}{\underline{|9}} - \ \ . \ . \tag{61}$$

and

$$\epsilon^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \frac{x^6}{\underline{|6}} + \ . \ . \ . \tag{62}$$

From Eq. (62), upon substituting $-x$ for $+x$ there results Eq. (63). If jx be substituted for x in Eq. (62), Eq. (64) is the result, while the substitution of $-jx$ for x gives rise to Eq. (65). Accordingly,

$$\epsilon^{-x} = 1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} - \frac{x^5}{\underline{|5}} + \frac{x^6}{\underline{|6}} - \ . \ . \ . \tag{63}$$

$$\epsilon^{jx} = 1 + jx - \frac{x^2}{\underline{|2}} - \frac{jx^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{jx^5}{\underline{|5}} - \frac{x^6}{\underline{|6}} - \frac{jx^7}{\underline{|7}} + \cdots \quad (64)$$

$$\epsilon^{-jx} = 1 - jx - \frac{x^2}{\underline{|2}} + \frac{jx^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} - \frac{jx^5}{\underline{|5}} - \frac{x^6}{\underline{|6}} + \frac{jx^7}{\underline{|7}} + \cdots \quad (65)$$

Multiplying Eq. (61) by j and adding to Eq. (60),

$$\cos x + j \sin x = 1 + jx - \frac{x^2}{\underline{|2}} - \frac{jx^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{jx^5}{\underline{|5}} - \frac{x^6}{\underline{|6}} - \frac{jx^7}{\underline{|7}} + \cdots \quad (66)$$

The right-hand member of Eq. (66) is seen to be identical with the right-hand member of Eq. (64), from which follows the familiar relation

$$\epsilon^{jx} = \cos x + j \sin x \quad (67)$$

Similarly, multiplying Eq. (61) by j and subtracting from Eq. (60) yield results identical with the right hand member of Eq. (65), whence

$$\epsilon^{-jx} = \cos x - j \sin x. \quad (68)$$

The series for the hyperbolic sines and cosines of a variable may also be readily derived from Eqs. (62) and (63); for, since

$$\cosh x = \frac{\epsilon^x + \epsilon^{-x}}{2}$$

and

$$\sinh x = \frac{\epsilon^x - \epsilon^{-x}}{2}$$

the series for cosh x is obtained by adding Eqs. (62) and (63) and dividing the result by 2, while the series for sinh x results from taking one-half the difference of Eqs. (62) and (63).

Thus

$$\cosh x = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \frac{x^6}{\underline{|6}} + \frac{x^8}{\underline{|8}} + \cdots \quad (69)$$

and

$$\sinh x = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \frac{x^7}{\underline{|7}} + \frac{x^9}{\underline{|9}} + \cdots \quad (70)$$

These series are rapidly convergent and are useful for computing tables of hyperbolic sines and cosines of real angles. Abbreviated tables of $\sinh \frac{x}{c}$ and $\cosh \frac{x}{c}$ are found in Appendix B.

Exponentials ϵ^{jx} and ϵ^{-jx} as Operators.—It has already been shown by Eqs. (67) and (68) that

$$\epsilon^{jx} = \cos x + j \sin x$$

and

$$\epsilon^{-jx} = \cos x - j \sin x.$$

Herefrom it is apparent that exponentials with imaginary exponents represent unit length vectors whose phase positions shift with the variable circular angle x. The truth of this statement is clearly evident from the polar expressions for the vectors. Thus,

$$\begin{aligned} \epsilon^{jx} &= \sqrt{\cos^2 x + \sin^2 x}\underline{/x} \\ &= 1\underline{/x}. \end{aligned}$$

Similiarly,

$$\epsilon^{-jx} = 1\underline{/-x}.$$

These unit vectors are shown in Fig. 12.

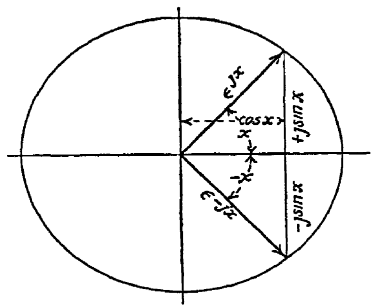

FIG. 12.—The unit vectors ϵ^{jx} and ϵ^{-jx}.

The length of the vector is constant regardless of the value of the angle, but, with increasing values of x, its slope increases in a counter-clockwise direction for the exponential ϵ^{jx} and in a clockwise direction for ϵ^{-jx}. If the angle x be made a function of time, such that $x = \omega t$, where ω is the angular velocity in radians per second, and t is the elapsed time in seconds, the unit vector will rotate at the constant angular velocity ω.

Considered as operators, then, the exponentials ϵ^{jx} and ϵ^{-jx} do not alter the size of the operand but simply change its slope. For the former, the shift is positive and, for the latter, negative through an angle of x circular radians.

Exponentials ϵ^{x} and ϵ^{-x} as Operators.—The curves

$$y. = \epsilon^{x}$$

and

$$y' = \epsilon^{-x}$$

are called exponential curves. When put in the form

$$x = \log_\epsilon y = \int \frac{dy}{y}$$

and

$$-x = \log_\epsilon y' = \int \frac{dy'}{y'}$$

they are called logarithmic curves, although the distinction is of little value. It should be noted, however, that these equations are similar in form to those given on page 15, that x is a real

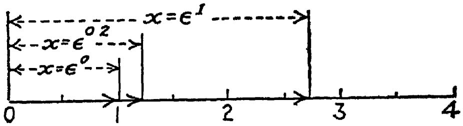

FIG 13.—The exponentials ϵ^x and ϵ^{-x}.

hyperbolic angle, and that the functions are unity for $x = 0$. The functions y and y' are vectors whose phase angles are always zero, but whose lengths vary with the hyperbolic angle x. Since

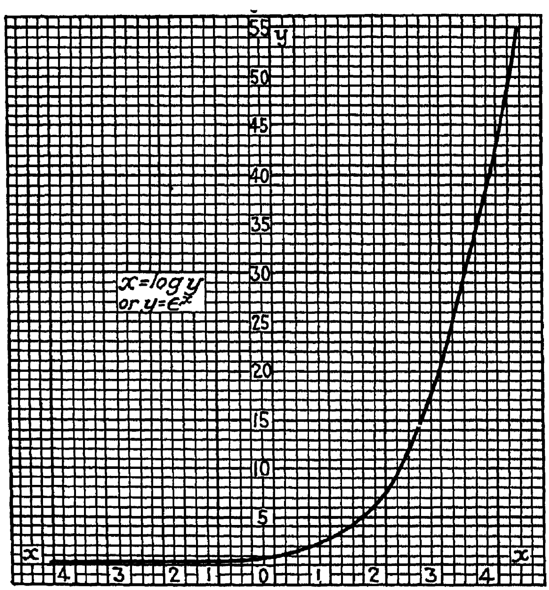

FIG. 14.—The graph of $y = \epsilon^x$.

the angle has no imaginary component, the vector does not shift its phase position as the angle increases, but can only stretch or shrink; that is, the end of the radius vector travels at all times in a straight-line path in the direction of its length, as in Fig. 13.

For positive values of the angle beginning with zero angle, the function ϵ^x increases from unity towards infinity at an accelerating rate or in geometric progression, as the angle increases. On the other hand, the function ϵ^{-x} starts at unity and, as the angle increases, diminishes or shrinks towards zero at a negatively accelerated rate. The graphs of these functions are shown in Fig 14.

Considered as operators, the effect of the exponential ϵ^x is to stretch or increase the operand through the hyperbolic angle x without shifting its phase position, while that of the operator ϵ^{-x} is to cause a shrinking of the operand, but also without shifting its phase position.

Exponential, Trigonometric and Hyperbolic Functions Related. That exponential, circular and hyperbolic functions are intimately related is evident from the foregoing. Any function of a variable expressed in terms of one of these may be converted into an equivalent expression in which only one or both of the other two appear. Some of the simpler of these relations are obtained as follows:

By the addition of Eqs. (67) and (68),

$$2 \cos x = \epsilon^{jx} + \epsilon^{-jx}$$

or

$$\cos x = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2}. \tag{71}$$

By subtraction of Eq. (68) from Eq. (67),

$$2\,j \sin x = \epsilon^{jx} - \epsilon^{-jx}$$

or

$$\sin x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{2j}. \tag{72}$$

By division of Eq. (72) by Eq. (71),

$$\tan x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{j(\epsilon^{jx} + \epsilon^{-jx})}. \tag{73}$$

Likewise, by substituting $x = jy$ in Eqs. (67) and (68), there results:

$$\epsilon^{-y} = \cos jy + j \sin jy \tag{74}$$

$$\epsilon^{y} = \cos jy - j \sin jy. \tag{75}$$

By adding Eqs. (74) and (75), it follows that

$$2 \cos jy = \epsilon^{y} + \epsilon^{-y}$$

or

$$\cos jy = \frac{\epsilon^{y} - \epsilon^{-y}}{2}. \tag{76}$$

Similarly, by subtracting Eqs. (74) and (75),

$$\sin jy = j\frac{(\epsilon^{y} - \epsilon^{-y})}{2} \tag{77}$$

and, by dividing Eq. (77) by Eq. (76),

$$\tan jy = j\left(\frac{\epsilon^{y} - \epsilon^{-y}}{\epsilon^{y} + \epsilon^{-y}}\right). \tag{78}$$

From the above, it is seen that exponential functions with imaginary angles may be expressed as trigonometric functions with real angles, or as hyperbolic functions with imaginary angles, while exponential functions with real angles will yield trigonometric functions with imaginary angles or hyperbolic functions with real angles. For convenient reference the relations here developed are assembled in Table 1.

TABLE 1

Exponentials with imaginary angles	Exponentials with real angles
$\epsilon^{jx} = \cos x + j \sin x$	$\epsilon^{y} = \cos jy - j \sin jy$
$\epsilon^{-jx} = \cos x - j \sin x$	$\epsilon^{-y} = \cos jy + j \sin jy$
$\sin x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{j2} = -j \sinh jx$	$\sin jy = j\frac{(\epsilon^{y} - \epsilon^{-y})}{2} = j \sinh y$
$\cos x = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2} = \cosh jx$	$\cos jy = \frac{\epsilon^{y} + \epsilon^{-y}}{2} = \cosh y$
$\tan x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{j(\epsilon^{jx} + \epsilon^{-jx})} = -j \tanh jx$	$\tan jy = j\frac{(\epsilon^{y} - \epsilon^{-y})}{(\epsilon^{y} + \epsilon^{-y})} = j \tanh y$

These relations among circular and hyperbolic functions make it possible to develop the formulas of hyperbolic trigonometry from those of circular trigonometry by simple substitution. For each one of the former there is one of the latter. For example, in circular trigonometry,

$$\sin^2 x + \cos^2 x = 1.$$

Then, if $y = jx$, substituting $j \sinh y = \sin x$ and $\cosh y = \cos x$ there follows:

$$j^2 \sinh^2 y + \cosh^2 y = 1$$

or

$$\cosh^2 y - \sinh^2 y = 1 \tag{79}$$

which is the corresponding identity in hyperbolic trigonometry. Again, from the trigonometry of circular functions,

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta.$$

By substituting

$$\begin{aligned} -j \sinh\gamma &= \sin\alpha \\ \cosh\gamma &= \cos\alpha \\ -j \sinh\delta &= \sin\beta \\ \cosh\delta &= \cos\beta \end{aligned}$$

there follows the equivalent hyperbolic form, namely,

$$\begin{aligned} \cosh(\gamma \pm \delta) &= \cosh\gamma \cosh\delta \pm j^2 \sinh\gamma \sinh\delta \\ &= \cosh\gamma \cosh\delta \pm \sinh\gamma \sinh\delta. \end{aligned} \tag{80}$$

In a similar manner,

$$\sinh(\gamma \pm \delta) = \sinh\gamma \cosh\delta \pm \cosh\gamma \sinh\delta. \tag{81}$$

By the same general method of substitution, any relation of circular trigonometry will yield the corresponding relation of hyperbolic trigonometry. In Appendix A some of the more commonly used hyperbolic equations are given.

Complex Angles.—Consistent with the definition of a circular angle as given in Eq. (37), the general or complex angle is defined as the length of arc per unit of radius, *i.e.*,

$$d\dot{\theta} = \frac{ds}{r}$$

or

$$\dot{\theta} = \int \frac{ds}{r}.$$

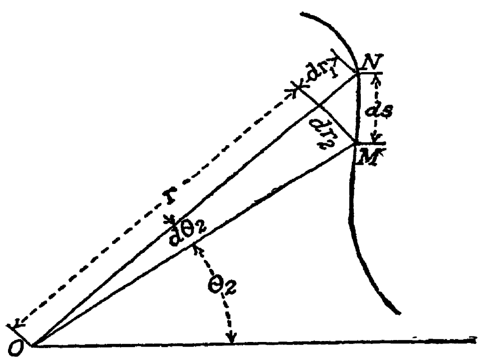

Fig. 15.—The complex angle.

then, considering the angular change produced when the radius vector traces along any curve such as MN (Fig. 15), thereby sweeping out a circular sector corresponding to a circular angle $d\theta_2$ circular radians, it may be noted that: (a) The radius vector changed its position with respect to the X-axis from θ_2 to $\theta_2 + d\theta_2$, and had it remained fixed in length during this change, it would have traced the arc $rd\theta_2 = dr_2$; (b) The radius vector changed in length

or stretched from r to $r + dr_1$, and, in so doing, had the angle θ_2 not changed in value, the former would have traced along the line dr_1.

The total change ds, in arc from M to N, may therefore be considered as having been accomplished by two steps in quadrature with each other. Considering the direction of the radius vector as the axis of reference, *i.e.*, the axis of reals, and taking the original length as the unit of measure, the vector change of arc may be denoted by the equation

$$d\dot{s} = dr_1 + jdr_2 \tag{82}$$

or, dividing both sides of Eq. (82) by r,

$$d\dot{\theta} = \frac{dr_1}{r} + j\,\frac{dr_2}{r}$$

and

$$\dot{\theta} = \int_1^r \frac{dr_1}{r} + j\int_0^s \frac{dr_2}{r}$$

or

$$\begin{aligned}\dot{\theta} &= \theta_1 + j\theta_2 \qquad (83)\\ &= \text{(hyperbolic angle)} + j\ \text{(circular angle)},\end{aligned}$$

in conformity with the definitions already given for these angles.[1]

The angle θ is a general or complex angle. Its real component is an hyperbolic angle θ_1, and its quadrature component the circular angle θ_2. The hyperbolic component measures the integrated percentage of the stretch of the radius vector while elongating or contracting, while the circular component measures the percentage of the arc or swing of the radius vector of fixed length.

Functions of Complex Angles.—Hyperbolic functions of complex angles may be taken from charts[2] or they may be computed from appropriate trigonometric relations. It will be sufficient for the object in view to show how to derive the values of hyperbolic sines and cosines of complex angles, since the other functions may readily be obtained if these are known.

[1] For a more complete discussion of complex angles see: BOYAJIAN, ARAM, "Complex Angles," *Jour.*, A. I. E. E., February, 1923; KENNELLY, A. E., "Hyperbolic Functions Applied to Electrical Engineering."

[2] KENNELLY, A. E., "Chart Atlas of Complex Hyperbolic and Circular Functions," Cambridge University Press, 1914.

By Eq. (80), the cosh of a complex angle is the complex quantity

$$\dot{A} = \cosh(x + jy) = \cosh x \cosh jy \pm \sinh x \sinh jy \tag{84}$$

$$= \cosh x \cos y + j \sinh x \sin y \tag{85}$$

$$A = \sqrt{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}$$

$$= \sqrt{\sinh^2 x + \cos^2 y}$$

or

$$= \sqrt{\cosh^2 x - \sin^2 y} \tag{86}$$

and

$$\theta_A = \tan^{-1}\left(\frac{\sinh x \sin y}{\cosh x \cos y}\right)$$

$$= \tan^{-1}(\tanh x \tan y). \tag{87}$$

Likewise, by Eq. (81), the sinh of a complex angle is the complex quantity

$$\dot{B} = \sinh(x \pm jy) = \sinh x \cosh jy + \cosh x \sinh jy \tag{88}$$

$$= \sinh x \cos y \pm j \cosh x \sin y \tag{89}$$

$$B = \sqrt{\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y}$$

$$= \sqrt{\sinh^2 x + \sin^2 y} \tag{90}$$

or

$$= \sqrt{\cosh^2 x - \cos^2 y} \tag{91}$$

and

$$\theta_B = \tan^{-1}\left(\frac{\cosh x \sin y}{\sinh x \cos y}\right)$$

$$= \tan^{-1}(\coth x \tan y). \tag{92}$$

Dividing Eq (89) by Eq (85) and simplifying yields

$$\dot{C} = \tanh(x \pm jy) = \frac{\sinh 2x \pm j \sin 2y}{\cosh 2x + \cos 2y} \tag{93}$$

With the help of Eqs. (86), (87), (91) and (92), together with tables of hyperbolic and circular functions of real angles, the hyperbolic functions of complex angles may be readily calculated.[1] Circular functions of complex angles may be calculated by the same general methods.

[1] The modern computing machine makes it very easy to perform lengthy calculations such as are required here, at the same time giving results to almost any desired accuracy. Such machines are now usually available in engineering offices. When they are available, tables of natural hyperbolic and circular functions are conveniently used. When not available, tables containing the logs of these functions will be found far more convenient. Very satisfactory tables, containing both the natural functions and their

Thus, if

$$\dot{P} = \sin\,(x + jy) = \sin x \cos jy + \cos x \sin jy \tag{94}$$

by substituting

$$\cos jy = \cosh y$$
$$\sin jy = j \sinh y$$

in Eq. (94), it follows that

$$\dot{P} = \sin x \cosh y + j \cos x \sinh y \tag{95}$$

and

$$P = \sqrt{\sinh^2 y + \sin^2 x}\,\underline{/\tan^{-1} \cot x \tanh y} \tag{96}$$

or

$$= \sqrt{\cosh^2 y - \cos^2 x}\,\underline{/\tan^{-1} \cot x \tanh y}. \tag{97}$$

Likewise, let

$$\dot{Q} = \cos\,(x \dotplus jy) = \cos x \cos jy \mp \sin x \sin jy. \tag{98}$$

Then, by transformation, as above,

$$\dot{Q} = \cos x \cosh y \mp j \sin x \sinh y \tag{99}$$
$$Q = \sqrt{\sinh^2 y + \cos^2 x}\,\underline{/\tan^{-1}(\tan x \tanh y)} \tag{100}$$
$$= \sqrt{\cosh^2 y - \sin^2 x}\,\underline{/\tan^{-1}(\tan x \tanh y)}. \tag{101}$$

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logs, are the Smithsonian Mathematical Tables published in 1909 by the Smithsonian Institute of Washington, D C. A very excellent table of the logs of hyperbolic functions is contained in the appendix to PERNOT'S "Electrical Phenomena in Parallel Conductors," vol. 1, John Wiley & Sons, Inc.

CHAPTER II

PROPERTIES OF TRANSMISSION-LINE CONDUCTORS

The important characteristics of transmission-line conductors are high electrical conductivity, high tensile strength, low density and low cost. The three metals that possess these characteristics to such a degree as to make their use economical in transmission-line construction are copper, aluminum and iron (steel). They are used alone or in various combinations. Of these, copper has the highest conductivity, is second in tensile strength and has the greatest density. Aluminum is second in conductivity, has the lowest tensile strength and the lowest density. Steel has the greatest strength, the lowest conductivity and is second in density. It has the additional disadvantage of a high internal reactance, is subject to hysteresis and eddy-current loss, and corrodes more readily than either of the other two.

Steel is used to particularly good advantage in combination with aluminum. Cables are built up with a steel core of either solid wire or stranded cable, depending upon the size of conductor. Thus the advantages of the high strength of steel and the light weight and high conductivity of aluminum are combined in a single cable. The use of steel reinforced cable makes it possible to use much longer and more economical spans than would be permissible with all-aluminum conductors.

Steel is also used in combination with copper in the so-called copper-clad steel conductors. These conductors are made from mild steel billets around which copper has been cast. In the casting process the steel billet is heated to a yellow heat so that the molten copper is welded firmly to it when poured around it in the mold. In the subsequent rolling and drawing processes the original relative thickness of the two metals is approximately maintained. The finished conductors are made in two commercial grades as 30 and 40 per cent of the volume conductivity of copper. Unless the steel core becomes exposed the conductor

does not corrode. Its strength depends upon the quality of steel used, and runs from $\frac{1}{3}$ to $\frac{2}{3}$ greater than for copper.

Steel cable has been used in transmission-line work chiefly as guard cables and for guying and anchoring in wood-pole construction. To a very limited extent it has also been used for short, relatively unimportant, lightly loaded transmission circuits, and, in at least one case, as a long span link in an important line. The Narrows Crossing on the city of Tacoma's Cushman line is a span of 6,244.5 ft., at present the longest transmission span in the world. The six cables of this span are made of 1¼ in. double-galvanized plow steel, having a tensile strength of about four times that of mild steel.

Steel cables and wires, however used in transmission-line work, whether alone or as cores in bi-metallic conductors, are heavily galvanized to prevent corrosion. Galvanizing softens the conductor somewhat and thereby reduces its tensile strength.

Density.—The density of a material is its weight per unit of volume. The densities of copper and aluminum vary only very slightly from the values given below. The density of iron in its various forms changes somewhat more depending upon its degree of chemical purity and its physical state.

Aluminum . . .	2 70 g	per cubic centimeter =	168 5 lb	per cubic foot
Copper.	8 89 g	per cubic centimeter =	555 lb	per cubic foot
Steel and iron.	7 86 g	per cubic centimeter =	490 lb.	per cubic foot

Conductivity and Resistivity.—The standard (100 per cent conductivity), to which electrical conductivities are compared, is known as the International Annealed Copper Standard. This standard copper has a density of 8.89 g. per cubic centimeter and a resistivity of 10.371 ohms per mil foot at 20° C. The conductivity of commercial grades of copper vary with the degree of hardness to which the conductors are drawn, and therefore with the size of the wire. In the sizes and in the degree of hardness used in medium hard-drawn, stranded cables, the conductivity of copper at 20° C. is about 97 per cent. The conductivity of hard-drawn aluminum is about 61 per cent, whence its resistivity is 17.01 ohms per mil foot at 20° C. The resistivity of steel wire varies considerably with its degree of hardness. Roughly, for hard, high-strength steel wires, it is ten times that of copper, while for iron wire it is from 6.5 to 8.5 times as great.

The resistances of stranded conductors are 2 per cent greater than the resistances of the equivalent solid conductors. This allowance is made to correct for the added length due to the "lay" of the cables, as recommended by the A.I.E.E.

Temperature Coefficient of Resistance.—This coefficient represents the change in resistance of a conductor in ohms per degree Centigrade per ohm of resistance. For 100 per cent conductivity copper, this constant is 0.00393 ohm per degree Centigrade at 20° C., while, for 97 per cent conductivity and the same temperature, it is 0.00381. The constant changes with both the temperature of reference and with the percentage of conductivity, it being proportional to the latter. For aluminum the average value of the coefficient is given by the Bureau of Standards as 0.00390 per degree Centigrade at 20°. This is also the value quoted by the Aluminum Company of America.

For 40 per cent copper-clad steel wire the coefficient varies from 0.004 to 0.005.

For iron and steel wires the coefficient varies considerably depending upon the grade of iron or steel and upon its hardness. An idea of the values to be expected in stranded steel cable may be had from the following figures published by the Indiana Steel and Wire Company

TABLE 2

Designation	Temperature coefficient per degree Centigrade at 20°
"High-strength strand"	0 00338
$\frac{1}{2}$-in. Siemens-Martin strand	0 00338
$\frac{3}{8}$-in. Siemens-Martin strand	0 00348
$\frac{1}{4}$-in. Siemens-Martin strand	0 00309
$\frac{3}{8}$-in. Standard strand	0 00570
No. 8 B.W G. "3-ply twisted guy wire"	0 00445
No 6 B.W.G. "B.B Telephone and Telegraph wire"	0.00496

The resistivity of metallic conductors at any temperature other than the base temperature is given by the equation

$$\rho_t = \rho(1 + \alpha t) \tag{102}$$

where

ρ_t = resistivity at temperature $t°$ C.
ρ = resistivity at base temperature
α = temperature coefficient of resistance in degrees C.
t = the rise in temperature above the base in degrees C. (sign of α is + for rise and − for fall).

Coefficient of Linear Expansion.—This coefficient is a measure of the change in length which accompanies a change of 1° in temperature, expressed as a decimal fraction. The following are average values, per degree Fahrenheit.

TABLE 3

Conductor material	Temperature coefficient of expansion, degrees Fahrenheit
Aluminum	12.8×10^{-6}
Aluminum-steel cable	10 to 11×10^{-6} (depending upon the ratio of steel to aluminum)
Copper	9.22×10^{-6}
40 per cent copper-clad steel	12.9×10^{-6}
Steel	6.4×10^{-6}

The corresponding values of the coefficients per degree Centigrade are obtained by multiplying the values in the table by the ratio 9 ÷ 5.

Tensile Strength.—The tensile strengths of aluminum, copper and steel depend considerably upon the degree of hardness to which they are drawn and upon the heat treatment received. Strength increases with the hardness, and, accordingly, is greater for the smaller sizes of wire than for the larger ones owing to the greater number of passes required in drawing.

The strength of concentric-strand cables is not equal to the sum of the strengths of its component strands, but is usually estimated to be 90 per cent of this value for copper and from 85 to 90 per cent of it for aluminum.

Tensile strengths of commercial grades of wire are approximately as given in Table 4.

TABLE 4

Conductor material and grade	Ultimate tensile strength (pounds per square inch)
Aluminum wire, hard drawn:	
a Sizes used as solid line conductors	22,000 to 24,000
b. Sizes used in stranded cables	24,000 to 27,000
Copper wire:	
a. Annealed	
Large sizes...	32,000 to 34,000
Small sizes	Up to 40,000
b. Medium hard drawn:	
Large sizes	40,000 to 50,000
Small sizes...	Up to 60,000
c Hard drawn:	
Large sizes	50,000
Small sizes	65,000
Steel wire	Varies greatly depending upon chemical constituency and hardness The range covered by the different available grades is from 50,000 to about 400,000

Elastic Limit.—A body is said to be elastic when, if subjected to a *strain*, internal *stresses* are set up which will bring the body back to its original form when the strain is removed. A cable under a moderate tension will stretch slightly, but, upon removing the tension, the original length of the cable will be resumed. If the tension on the cable is gradually increased, some value of tension will finally be reached beyond which the cable will not resume its original length when the strain is removed; instead, the cable will be permanently deformed. The lowest tension at which such deformation occurs is called the "elastic limit" of the cable.

The elastic limit is not a very definite point in certain soft forms of conductor materials such as annealed copper and aluminum, since these materials begin to stretch at relatively low tensions. The term "elastic limit," as applied to these materials, has a somewhat special meaning not strictly in accord with the above definition.

The elastic limit of hard-drawn aluminum wire runs from 50 to 60 per cent of its ultimate tensile strength or from about 12,000 to 16,000 lb. per square inch. The value quoted by the Aluminum

Company of America is 14,000 lb. Steel-reinforced aluminum cables have an elastic limit equal to about 65 per cent of the ultimate strength of the cable.

Hard-drawn copper has an elastic limit of from 50 to 65 per cent of its ultimate strength, depending upon the degree of hardness; 50 per cent is a conservative figure that is much used.

The elastic limit of commercial steel wire is about 50 per cent of its ultimate strength, while for special high-strength steels it is much higher. For the steels used in reinforced-aluminum cables, the Aluminum Company of America gives the ultimate strength as 160,000 and the elastic limit as 130,000 lb. per square inch, respectively.

In the design of transmission line spans the question arises as to what maximum allowable tension is permissible for a given conductor. Here engineering practice has usually been to assume this to be from 75 to 100 per cent of the elastic limit of the cable. Any load slightly in excess of the elastic limit will cause the cable to stretch slightly, thus relieving the tension and increasing the sag. The length and sag at any given temperature will be permanently increased, but the strength of the cable will be unimpaired.

Modulus of Elasticity.—So long as the elastic limit of a material is not exceeded, the strain or deformation produced is proportional to the stress applied. This is known as Hooke's law. Under these conditions, the ratio of stress per unit area to the deformation per unit of length is a constant amount E for a given piece of material. This ratio is called Young's modulus of elasticity.

Thus, consider a wire or cable having A sq. in. of cross-sectional area and a length of L_0 in. when subjected to an initial tension of T_0 lb. If the wire is subsequently stressed to a slightly greater tension T lb., thereby producing an additional elongation of dl in. by Hooke's law (assuming the change in area due to the elongation to be a negligible amount),

$$E = \frac{T - T_0}{A} \div \frac{dl}{L_0} = \frac{(T - T_0)L_0}{A \cdot dl}. \qquad (103)$$

The elongation is

$$dl = \frac{(T - T_0)L_0}{EA}$$

and the final length of the cable is

$$
\begin{aligned}
L &= L_0 + dl \\
&= L_0\left[1 + \frac{(T - T_0)}{EA}\right]
\end{aligned}
\tag{104}
$$

It is difficult to determine experimentally the exact values of the modulus of elasticity for materials like copper and aluminum because they have no clearly defined elastic limits. These materials do not obey Hooke's law closely within the elastic limit, as a perfectly elastic material does. Furthermore, since stressing such materials tends to increase their tensile strengths, the modulus is usually higher after stressing than before. The modulus is less for stranded cables than for solid wires, and less for annealed than for hard-drawn wires. The values given below are average values of E in pounds per square inch for the various materials and grades:

TABLE 5

Material	Young's modulus, pounds per square inch
Aluminum wires and cables	9×10^6
Aluminum (steel reinforced)	11.4×10^6 to 13×10^6
Copper wire (annealed)	12×10^6
Copper wire (hard-drawn)	16×10^6
Copper concentric-strand cable	About 75 per cent of the corresponding solid wire
Copper-clad steel cable	16×10^6 to 20×10^6
Iron wire	24×10^6
Steel wire	27×10^6
Steel concentric lay cable	22×10^6

CHAPTER III

THE MAGNETIC CIRCUIT AND INDUCTANCE

Any space in which a magnet pole experiences magnetic forces is a magnetic field. When dealing with these fields and the magnetic forces associated with them, our thinking is greatly facilitated by the use of Faraday's concept of *magnetic lines of force.* These lines are so drawn that their directions at all points represent the directions of the resultant magnetic forces, and their number per unit area normal to their direction—that is, their density, in a field of unit permeability—represents the intensity of the forces acting.

Since the basic theory of the transmission-line circuit is intimately concerned with the magnetic circuit calculations, it will be advantageous, at this point, to discuss briefly the fundamental relations and the units involved.

Concepts, Definitions and Units.—Unless otherwise stated, the units used in the following definitions are the *absolute electromagnetic units.*

Measurements of magnetic field intensity are based upon the concept of an ideal *unit point magnetic pole.* Such a unit pole is a pole which in a medium of air, when separated from an equal pole of like sign by a distance of one centimeter, is repelled with a force of one dyne.

The *magnetic potential* of a point is the number of ergs of work done in bringing a unit point pole from the edge of the field to the point in question.

The *intensity* of a magnetic field is the force in dynes which unit point north pole experiences in the field, and is designated by the symbol H. Since the direction of a magnetic line of force is also the direction of the magnetic field intensity, if distances along a path be l, then the *magnetic potential difference* between any two points in a field of force is the line integral of the magnetic field intensity; that is,

$$dF = H \quad dl \cdot \cos \theta$$

and

$$F = \int_{(L)} H \cdot dl \cdot \cos\theta \tag{105}$$

where

H = the magnetic potential gradient in the direction of the field
dl = an elementary length of path

and

θ = the angle between the direction of the intensity H and dl.

The c g.s. unit of field intensity is the *gilbert per centimeter*. *The ampere-turn per inch* is the common practical unit. The two units are related by the equation

$$NI(\text{per inch}) = 2.022H \text{ (per centimeter)} \tag{106}$$

where I is the current measured in amperes.

The drop of magnetic potential between any two points on a line of force in a magnetic circuit is the *magnetomotive force* F, absorbed in that portion of the circuit, as given by Eq. (107) below. Here again the current is given in amperes.

$$F = Hl = 0.4\pi NI \tag{107}$$

The c.g s. unit of m.m.f. is the gilbert and the practical unit is the *ampere-turn*.

The unit of magnetic flux is a *line of force* called the "maxwell." The total number of maxwells in a given sectional area is represented by the symbol ϕ, while the *flux density* is $B = \frac{\phi}{A}$, where A is the sectional area of the path in square centimeters, normal to the direction of the flux. The unit of flux density, the *gauss*, represents 1 line per square centimeter.

In a medium of air a field intensity of 1 gilbert per centimeter will establish a flux density of 1 gauss, since the magnetic conductivity of air per cubic centimeter is unity. Magnetic materials have very much higher magnetic conductivities. In them the same field intensity will establish a flux having a density of many gausses. The ratio of flux density to field intensity in a material; $\mu = \frac{B}{H}$, is called the "permeability" of the material.

The law of the magnetic circuit, analogous to Ohm's law, and expressing the relation of cause and effect for the magnetic circuit, is

$$\phi = \frac{F}{R}$$
$$= \frac{4\pi NI}{\frac{l}{\mu A}}. \qquad (108)$$

The quantity designated by R is called the *reluctance* of the magnetic circuit.

When a magnetic field is produced by a current, variations in the current are accompanied by variations in the magnetic flux. The varying flux induces in the conductor a voltage which tends to prevent the change in current that causes it. These relations are expressed symbolically in c.g.s. units by the equation

$$e = -L\frac{di}{dt} = -N\frac{d\phi}{dt} \qquad (109)$$

where L is the proportionality factor between the time rate of change of current and the induced potential difference. From Eq. (109) it follows that

$$L = N\frac{d\phi}{di}$$

or, if the permeability of the magnetic circuit is constant,

$$L = \frac{N\phi}{I}. \qquad (110)$$

The product $N\phi$ represents the total number of *magnetic linkages* which the current I produces in the circuit. Thus, the c.g.s. unit of L, called the *coefficient of self-inductance* of a circuit, is the number of magnetic linkages per absampere of current. This unit is the *abhenry*. The practical unit, called the "henry," is 10^9 times as large as the abhenry.

By multiplying Eq. (109) through by idt and integrating, the energy stored in the magnetic field when the current in the circuit is I units is found to be

$$E = \frac{LI^2}{2} \text{ joules} \qquad (111)$$

when L is given in henries, and I in amperes.

The Magnetic Field about a Long, Straight Cylinder.—Consider an elementary length dl of a long, straight, round conductor (Fig. 16). Let the conductor be of infinitesimal cross-section, and let the current in it be I absamperes. A unit north magnet pole at P will set up a flux density at O of $B = \frac{1}{x^2}$ gausses. The component of this flux density normal to the direction of the current is $B \cos \theta$, and is the component which reacts with the current in dl to produce a force. Due to it, the length dl of the conductor is pushed upwards with a force of $BI \cos \theta \cdot dl$ dynes. The unit pole at P is pushed downward with an equal force, due to the field at P caused by the current in dl. If the field intensity at P, due to dl, is dH, the force on the pole is dH.

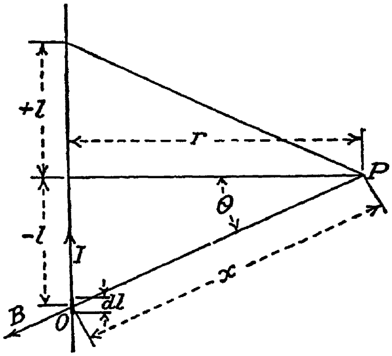


FIG 16—The magnetic field intensity near a long straight wire.

Expressing the equality of the forces on dl and on the pole at P, and substituting for B the value $\frac{1}{x^2}$, there results the equation,

$$dH = \frac{I \cos \theta \cdot dl}{x^2} \text{ gilberts per centimeter.} \tag{112}$$

From the figure, $x = r \sec \theta$ and $l = r \tan \theta$, whence $dl = r \sec^2 \theta d\theta$. By substituting the values of dl and x^2 in Eq. (112), this equation is transformed to

$$dH = \frac{I \cos \theta \cdot d\theta}{r}. \tag{113}$$

For a very long, straight wire, since θ approaches the value $\frac{\pi}{2}$, the field intensity due to the entire length of conductor, and at a distance of r cm. from it, is

$$H = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{I \cos \theta \cdot d\theta}{r}$$

$$= \frac{2I}{r} \text{ gilberts per centimeter.} \tag{114}$$

Thus, the field intensity at any point without a line conductor is inversely proportional to the distance from the center of the conductor. The same law holds for circular conductors, and practically also for conductors of other shapes when the distance of the point from the conductor is large as compared with the dimensions of the conductor area.

Magnetic Lines of Force about a Straight, Round Conductor.—It is to be remembered that field intensity is a vector quantity. Its magnitude is given by Eq. (114), while its direction and sense are found by the familiar right-hand rule. Lines of constant field intensities drawn about a straight, round conductor, in planes perpendicular to its length, are the familiar lines of force. By Eq. (114) these are concentric circles.

. The circular lines of force about a single isolated conductor (in a medium of constant permeability) are illustrated in Fig. 17.

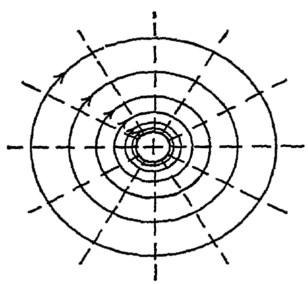


Fig. 17.—The magnetic field about an isolated current.

The lines are drawn close together near the conductor, but, as distances from the conductor increase, they are separated by increasing spaces to represent the diminishing field intensities. The total drop of magnetic potential along a line of force is the same for all lines, and is equal to the magnetomotive force of the current in the conductor; that is, for any line of force the line integral of the field intensity ($\int_L H \cdot dl \cdot \cos\,\theta$), taken completely around the circuit, is equal to $4\pi I$. For the line of force distant r cm. from a straight wire,

$$4\pi I = \int_0^{2\pi r} H \cdot dl. \tag{115}$$

This is essentially a statement of Kirchoff's law as applied to the magnetic circuit about the conductor.

It should be noted that in Fig. 17 the full lines are the circular lines of force. The resultant field intensity is tangent to these circles at every point. The broken lines are lines of constant magnetic potentials. (Dielectric lines of force as will be shown later.)

Magnetic Field Intensity within a Round, Straight Conductor. The field intensity at any point within a straight, round conductor having a uniform current density, is proportional to the distance from the center of the conductor. The proof of this

statement is apparent from the following: In Fig. 18, consider the field intensity at a point P distant x cm. from the center of a circular conductor of radius r cm. Let the current I in the conductor be uniformly distributed over its entire area. The field intensity at P is uninfluenced by the current in that part of the conductor lying without the circle of radius x. The current I_x lying within the inner circle is

$$I_x = \frac{Ix^2}{r^2}$$

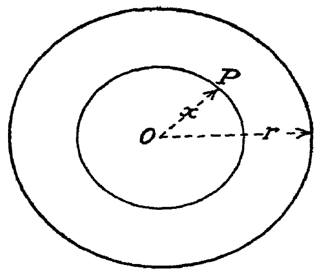


FIG. 18—The magnetic field intensity within a straight, round wire

and its magnetizing effect at P is the same as though it were all flowing along the axis of the conductor. The field intensity at P may then be found from Eq. (114). It is

$$H_p = \frac{2Ix}{r^2}. \tag{116}$$

In Fig. 19 the field intensities both inside and outside the conductor are shown as functions of distance from the axis of the conductor.

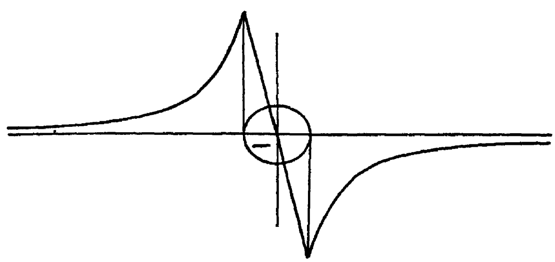


FIG 19—Curve of magnetic field intensity due to the current in a wire.

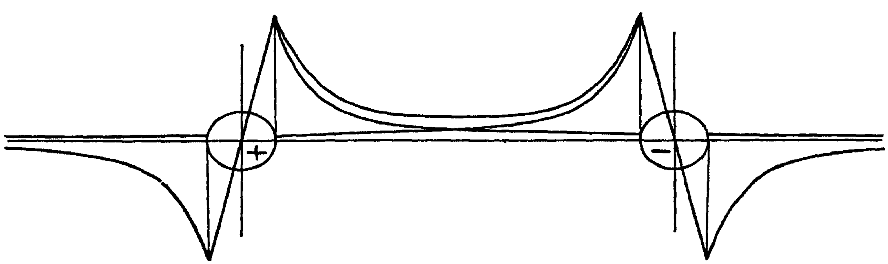


FIG 20.—The curve of magnetic field intensity due to the current in a parallel sided, return loop.

Figure 20 shows the separate field intensity curves due to each of two parallel, round conductors, carrying equal currents of

opposite sign, together with the resultant curve for the two conductors. The permeabilities of the conductors and the medium in which they are suspended are assumed to be unity.

Calculation of Magnetic Flux about a Straight, Round Conductor.—In transmission-line calculations, problems frequently arise in which it is necessary to compute the amount of magnetic flux passing between the two parallel sides of a loop, the sides of which are parallel to the conductors of the transmission line. The method employed in making these calculations may be illustrated by the following problem:

In Fig. 21, let a straight, round conductor at A be suspended in a medium of constant permeability μ. Assume the current of $+I$ amp. flowing in the conductor to be the only magnetizing force present. Let the current be uniformly distributed over the cross-section of the conductor. At all points without the conductor the magnetic effect is then the same as that produced by a current filament of $+I$ amp. coincident with the axis of the conductor. The problem is to find the number of magnetic lines of force per unit length of conductor, passing through any parallel-sided loop such as PQ, the sides of which are also parallel to the conductor A.

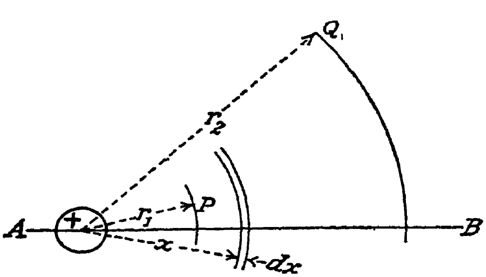


FIG 21 —Calculation of the magnetic flux through a loop.

It has already been shown that the magnetic lines of force due to the current are concentric circles in this case. Accordingly, between two unit length cylinders having radii r_1 and r_2, whose axes are coincident with the axis of the conductor, there will be included the lines of force whose total number we desire to know. Equation (114) shows that at any point on any one of the circular lines of force, such as the one of radius x cm., for example, if the current is measured in amperes, the field intensity has the constant value,

$$H = \frac{2I}{10x} \text{ gilberts per centimeter}$$

and the flux density is $B = \mu H$.

Since radial lines such as AB are lines of constant magnetic potential, the flux crosses them at right angles. The flux per

unit length of conductor, passing between the two concentric cylinders of radii x and $x + dx$, is, therefore,

$$d\phi = Bdx = \frac{2I\mu dx}{10x} \text{ lines} \tag{117}$$

and that passing between the concentric cylinders of radii r_1 and r_2 is

$$\phi_{PQ} = \frac{2I\mu}{10}\int_{r_1}^{r_2}\frac{dx}{x}$$

$$= \frac{2I\mu}{10} ln\frac{r_2}{r_1}. \tag{118}$$

The Magnetic Lines of Force about Two Parallel, Round Wires are Circles.—The magnetic lines of force due to the combined, equal magnetomotive forces of a parallel-sided return loop in a

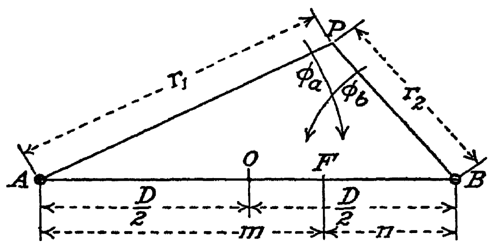


FIG 22 —The magnetic lines of force about the two round, parallel wires of a return loop circuit are circles

medium of constant permeability are circles. This fact may be proved as follows: In Fig. 22, let A and B be the two sides of such a loop carrying the equal currents $+I$ and $-I$ amp. as indicated. Let the point P move in the direction of the resultant magnetic field, that is, in a manner so that at every position of the point the drop of magnetic potential along a line at right angles to the direction of its motion is zero, and let some fixed point F on the line AB be on the locus of P. Since the locus of P is along the direction of the resultant field at every point, the resultant flux through the loop PF must be zero.

By the same line of reasoning as that used in deriving Eq. (118), the fluxes through this loop, due to the magnetomotive forces of the currents in A and B, are

$$\phi_a = \frac{2\mu I}{10} ln\frac{r_1}{m} \tag{119}$$

$$\phi_b = \frac{2\mu I}{10} ln\frac{r_2}{n}. \tag{120}$$

The resultant flux is

$$\phi_c = \phi_a + \phi_b$$
$$= \frac{2\mu I}{10}\left[ln\frac{r_1}{m} - ln\frac{r_2}{n}\right] = 0$$

whence

$$\frac{r_1}{m} = \frac{r_2}{n}$$

and

$$\frac{r_1}{r_2} = \frac{m}{n} = \text{a constant.} \tag{121}$$

Thus, since the point P moves so that the ratio $r_1 \div r_2$ of its distances from the two fixed points A and B is a constant, the locus of P is a circle by the theorem of the inverse points of a circle.

The Theorem of the Inverse Points of a Circle.—This theorem will now be demonstrated. In Fig. 23 let P be the point whose

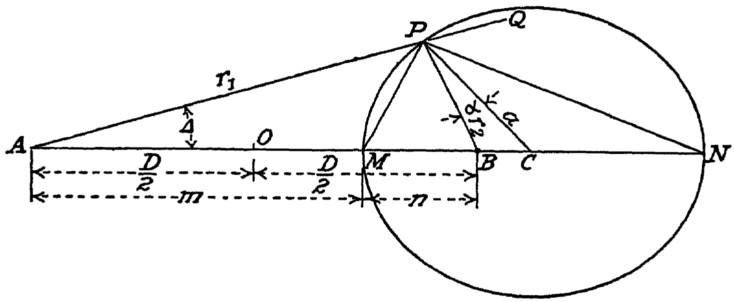


FIG 23.—The inverse points with respect to a circle.

distances r_1 and r_2 from the fixed points A and B, respectively, are in the constant ratio $\frac{m}{n}$. Bisect the angles APB and BPQ by the lines PM and PN. Since the bisector of any angle of a triangle divides the base into segments proportional to the adjacent sides, $\frac{AM}{MB} = \frac{m}{n} = \frac{NA}{NB}$. For any position of P the angle MPN is a right angle, since the external and internal bisectors of any angle of a triangle are perpendicular to each other. It follows that the locus of P is a circle, since the right triangle MPN has the fixed hypothenuse MN. The points A and B are called the inverse points with respect to the circle.

The following relations may also be noted: It is easily proven that angle BPC = angle PAB. The triangles CAP and CPB are

therefore similar, since the angles BPC and PAB are equal, and the angle PCB is common to both triangles. Hence their corresponding sides are proportional and

$$\frac{AC}{a} = \frac{r_1}{r_2} = \frac{a}{BC} = \frac{m}{n} = \text{a constant.} \tag{122}$$

Also

$$BC = \frac{a^2}{AC}. \tag{123}$$

Dipolar Circles.—In Fig. 24, the circles drawn in full lines are the magnetic lines of force, due to two parallel current filaments.

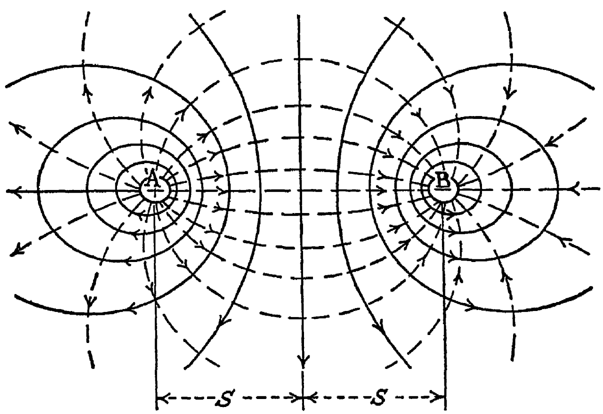


FIG 24.—The magnetic and dielectric fields about the two round wires of a parallel-sided return loop

These circles belong to two families, one described about A and the other about B, known as dipolar circles. The dotted circles are the corresponding lines of constant magnetic potentials. The latter are discussed in a later article.

Certain relations which are useful in drawing the circles are given below. These may readily be proved from Fig. 23. Let the constant ratio $\frac{m}{n} = e$, and let the spacing be $D = 2s$. Then, in Fig. 23,

$$\left.\begin{array}{ll} OM = \dfrac{s(1-e)}{1+e} & MN = \dfrac{4se}{1-e^2} \\[2ex] MB = \dfrac{2se}{1+e} & MC = \dfrac{2se}{1-e^2} \\[2ex] NB = \dfrac{2se}{1-e} & OC = \dfrac{s(1+e^2)}{1-e^2} \end{array}\right\}$$

Equation of the Circles.—Using rectangular coordinates with the origin at O, the equation of these circles is derived as follows: From Fig. 22, taking the distance between current filaments equal to $2s$, and letting x and y be the coordinates of P,

$$r_1^2 = (s + x)^2 + y^2 = s^2 + 2sx + x^2 + y^2$$
$$r_2^2 = (s - x)^2 + y^2 = s^2 - 2sx + x^2 + y^2$$

and, since

$$\frac{r_1^2}{r_2^2} = \frac{m^2}{n^2}$$

$$m^2(s^2 - 2sx + x^2 + y^2) = n^2(s^2 + 2sx + x^2 + y^2)$$

or

$$y^2(m^2 - n^2) + x^2(m^2 - n^2) - 2sx(m^2 + n^2) = -s^2(m^2 - n^2)$$

whence

$$y^2 + \left[x - \frac{m^2 + n^2}{2(m - n)}\right]^2 = \frac{m^2n^2}{(m - n)^2}. \qquad (124)$$

Equation (124) is the equation of a circle whose center is at

$$x = \frac{m^2 + n^2}{2(m - n)},$$
$$y = 0$$

and whose radius is

$$\rho = \frac{mn}{m - n}.$$

Lines of Equal, Magnetic Potentials of a Parallel-sided, Return Loop are Circles.—To prove this, let A and B of Fig. 25 be the two sides of a parallel-sided loop carrying the current filaments of strength $+I$ and $-I$. Let Q be a point which moves along a line of constant magnetic potential P, and let the fixed point F, lying on the bisector of AB, be a point on the locus of Q.

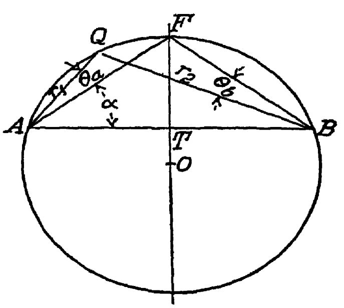


Fig. 25.—The lines of equal magnetic potentials of a parallel sided return loop.

Due to the currents at A and B each acting alone, the corresponding field intensities at Q, due to A and B, are

$$H_a = +\frac{2I}{r_1}$$
$$H_b = -\frac{2I}{r_2}.$$

The corresponding differences of magnetic potential between Q and F are

$$P_a = H_a r_1 \theta_a = 2I\theta_a$$
$$P_b = H_b r_2 \theta_b = -2I\theta_b.$$

Since Q and F are on an equipotential line,

$$P_a + P_b = 0 = 2I(\theta_a - \theta_b)$$

or

$$\theta_a = \theta_b. \tag{125}$$

If a circle be passed through the three points, A, F and B, the point Q will lie on this circle, for, since $\theta_a = \theta_b$ and both are angles inscribed in a given circle, they must intercept equal arcs. The circles of constant magnetic potentials and the circles representing magnetic lines of force are mutually perpendicular.

In the following chapter it will be shown that the circular magnetic lines of force are also the equipotential circles of the dielectric field, while the circles of equimagnetic potentials are the circular dielectric lines of force.

The Magnetic Flux-linkages of a Parallel-sided Return Loop. Coefficient of Self-inductance.—In Fig. 26, A and B represent the two sides of a parallel-sided return loop, carrying the currents of strength $+I$ and $-I$ absamperes, respectively. The medium is assumed to be air, having constant permeability equal to unity. Since the permeability is constant, the principle of superposition may be applied; that is, the flux through the loop AB may be found by adding the separate fluxes in the loop due to A and B, each calculated separately as though the other were not present.

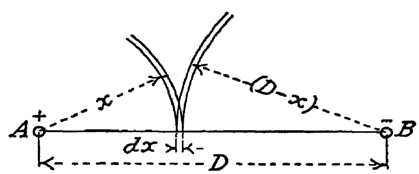


FIG 26.—Calculation of the magnetic flux linking a parallel-sided return loop.

The flux through the loop is evidently that which crosses the line AB. At a point on AB distant x cm. from A, the field intensities due to A and B have the values

$$\left.\begin{aligned} H_a &= \frac{2I}{x} \\ H_b &= \frac{2I}{D - x} \end{aligned}\right\} \tag{126}$$

Both of these intensities act downward through the loop at right angles to AB, and may therefore be added algebraically to find

the resultant intensity at the point considered. Also, since $\mu = 1$, $H = B$, and

$$B = 2I\left(\frac{1}{x} + \frac{1}{D - x}\right),$$

the flux between conductors crossing AB per centimeter length of line is

$$\begin{aligned} d\phi &= Bdx \\ &= 2I\left(\frac{1}{x} + \frac{1}{D - x}\right) \cdot dx \text{ lines.} \end{aligned}$$

The corresponding flux through the loop between conductors is

$$\begin{aligned} \phi &= 2I\int_{r}^{D-r}\left(\frac{1}{x} + \frac{1}{D - x}\right) dx \\ &= 4Iln\,\frac{D - r}{r} \text{ lines,} \end{aligned} \tag{127}$$

where r is the radius of the conductor in the units used to measure D. By definition, the coefficient of self-inductance is

$$L = \frac{N\phi}{I}$$

and, since $N = 1$, the inductance per centimeter of loop (not including the linkages within the conductors), is, from Eq. (127),

$$L_1 = 4ln\,\frac{D - r}{r} \text{ abhenries per centimeter of loop.} \tag{128}$$

Since only half of the flux of Eq. (127) links each conductor, the value of L_1 per centimeter length of conductor is

$$L_1 = 2ln\,\frac{D - r}{r} \text{ abhenries per centimeter of conductor.} \tag{129}$$

For all practical transmission lines no appreciable error is made by substituting D for $D - r$. Making this substitution, using the mile as the unit of length, $\log_{10}$ and the practical unit of inductance, the value of L_1, per mile of one conductor, is

$$L_1 = 741.13 \times 10^{-6} \log_{10} \frac{D}{r} \text{ henries per mile.} \tag{130}$$

On the assumption of uniform distribution of current over the cross-section of the conductor, the coefficient of self-inductance L_2, due to the flux-linkages within the conductor, is found as follows:

By Eq. (116) (Fig. 18), the field intensity at a point within the conductor and distant x cm. from the center, is

$$H_x = \frac{2Ix}{r^2}.$$

Accordingly, if the permeability of the conductor material is μ, the flux density corresponding to H_x is

$$B_x = \frac{2I\mu x}{r^2}$$

and the flux, per unit of length of conductor in a circular strip of width dx, becomes

$$d\phi = B_x dA = \frac{2I\mu x dx}{r^2}.$$

This flux links the current lying within the circle of radius x; that is, each flux line represents the fraction $\frac{x^2}{r^2}$ of a complete linkage with all the current. Thus,

$$d(N\phi) = \frac{2I\mu x}{r^2}\cdot\frac{x^2}{r^2}\cdot dx$$

whence

$$\begin{aligned} N\phi &= \frac{2I\mu}{r^4}\int_0^r x^3 dx \\ &= \frac{I\mu}{2} \text{ linkage} \end{aligned}$$

and

$$L_2 = \frac{N\phi}{I} = \frac{\mu}{2} \text{ abhenries per centimeter of conductor.} \qquad (131)$$

Converted to practical units per mile of one conductor,

$$L_2 = 80.47 \times 10^{-6}\mu \text{ henries per mile} \qquad (132)$$

The final expression for the total inductance, per mile of a single conductor, is the sum of Eq. (130) and (132), or

$$L = \left(741.13 \log_{10} \frac{D}{r} + 80.47\mu\right)10^{-6} \text{ henries per mile of one conductor.} \qquad (133)$$

Inductance of Split-conductor, Single-phase Circuits.—Consider a single-phase circuit as illustrated in Fig. 27, in which the outgoing current is divided between the equal paralleled conductors a and b. The same current returns through the corresponding equal paralleled conductors a' and b'. The radius of each conductor is r, and the spacings are D and kD, as shown, where k is a constant. The radii and spacings are measured in the same units.

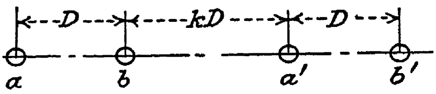


FIG. 27.—The split-conductor, single-phase circuit

If the total complex current in the circuit is I_0, and if a' and b' are the return conductors for a and b, respectively, then

$$I_0 = I_a + I_b = -(I_a' + I_b')$$

and

$$I_a = -I_a'$$
$$I_b = -\,I_b'.$$

Furthermore, since the outgoing conductors and the return conductors each constitute a pair of paralleled conductors, and since their cross-sections are equal and uniform,

$$Z_a I_a = Z_b I_b$$

and

$$Z_a' I_a' = Z_b' I_b'$$

where Z_a, Z_b, Z_a' and Z_b' are the complex impedances per centimeter of the conductors indicated by the subscripts. That is, in general,

$$ZI = (R + j\omega L)I$$

where R and ωL are respectively the resistance and the inductive reactance per unit length of conductor, carrying the complex current I.

The problem is to find the impedance drop in each of the pairs of conductors.

Referring to the figure, if S represents a very large but finite distance, the vector flux linking conductor a, due to all the currents, is

$$\phi_a = 2\left(I_a ln\frac{S}{r} + I_b ln\frac{S}{D} - I_a ln\frac{S}{D(1+k)} - I_b ln\frac{S}{D(2+k)}\right) + I_a\frac{\mu}{2}$$
$$= I_a\left(2ln\frac{D(1+k)}{r} + \frac{\mu}{2}\right) + 2I_b ln(2+k). \tag{134}$$

Similarly

$$\phi_b = 2\left(I_a ln\frac{S}{D} + I_b ln\frac{S}{r} - I_a ln\frac{S}{kD} - I_b ln\frac{S}{D(1+k)}\right) + \frac{I_b\mu}{2}$$
$$= I_b\left(2ln\frac{D(1+k)}{r} + \frac{\mu}{2}\right) + 2I_a lnk. \tag{135}$$

However,

$$\phi_a = L_a I_a$$

and hence

$$Z_a I_a = RI_a + j\omega\phi_a$$

and, similarly,

$$Z_b I_b = RI_b + j\omega\phi_b.$$

Accordingly, since a and b are in parallel,

$$RI_a + j\omega\left(2I_a ln\frac{D(1+k)}{r} + 2I_b ln(2+k) + \frac{I_a\mu}{2}\right) = RI_b$$
$$+ j\omega\left(2I_b ln\frac{D(1+k)}{r} + 2I_a lnk + I_b\frac{\mu}{2}\right). \tag{136}$$

Substituting $I_b = I_0 - I_a$ and solving for I_a,

$$I_a = -I_0\left(\frac{ln\frac{2+k}{k}}{\frac{2R}{j\omega} + \mu + 2ln\frac{D^2(1+k)^2}{r^2k(2+k)}} - \frac{1}{2}\right)\text{ vector amp.} \tag{137}$$

and

$$I_b = I_0\left(\frac{ln\frac{2+k}{k}}{\frac{2R}{j\omega} + \mu + 2ln\frac{D^2(1+k)^2}{r^2k(2+k)}} + \frac{1}{2}\right)\text{ vector amp.} \tag{138}$$

The impedance drop in each of the conductors may be found by substituting these values of current in the above equation for ZI_a or ZI_b, since the drop is the same in each of the two conductors.

That is,

$$ZI_a = I_0\left\{\left[R + j\omega\left(2ln\frac{D(1+k)}{r} + \frac{\mu}{2}\right)\right]\right.$$

$$\times\left[\frac{1}{2} - \frac{ln\frac{2+k}{k}}{\frac{2R}{j\omega} + \mu + 2ln\frac{D^2(1+k)^2}{r^2k(2+k)}}\right]$$

$$\left. + 2ln(2+k)\left[\frac{1}{2} + \frac{ln\frac{2+k}{k}}{\frac{2R}{j\omega} + \mu + 2ln\frac{D^2(1+k)^2}{r^2k(2+k)}}\right]\right\} \qquad (139)$$

This equation can be somewhat simplified. It will, however, serve the purpose about as well to leave the expression as it stands. Assuming I_0 to be known, the complex currents may be evaluated from Eq. (137) and (138), after which these values may readily be substituted in Eq. (139) to find the drop.

Inductance of Three-phase Lines. General Equations.—To illustrate the general method of calculating the drop in each conductor of a three-phase line, regardless of whether or not the currents are balanced, assume an untransposed three-phase line made up of three equal conductors, each of radius r, and having any spacings whatever, as in Fig. 28. Let S be a very large finite distance, and let the spacings, D_1, D_2 and D_3, as well as S and r, all be measured in the same units of length.

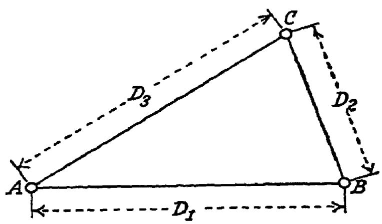


FIG. 28.—Triangular spacing for a three-phase line.

Then, in complex notation, the flux lines per centimeter, linking each of the three conductors, are the following:

$$\phi_a = 2\left(I_a ln\frac{S}{r} + I_b ln\frac{S}{D_1} + I_c ln\frac{S}{D_3}\right) + I_a\frac{\mu}{2} \qquad (140)$$

$$\phi_b = 2\left(I_a ln\frac{S}{D_1} + I_b ln\frac{S}{r} + I_c ln\frac{S}{D_2}\right) + I_b\frac{\mu}{2} \qquad (141)$$

$$\phi_c = 2\left(I_a ln\frac{S}{D_3} + I_b ln\frac{S}{D_2} + I_c\, ln\frac{S}{r}\right) + I_c\frac{\mu}{2}. \qquad (142)$$

The three complex currents are I_a, I_b and I_c, and their sum is zero. Thus,

$$I_a + I_b + I_c = 0.$$

If in Eq. (140), $-(I_a + I_b)$ be substituted for I_c, there results the relation

$$\phi_a = 2\left(I_a ln\frac{D_3}{r} + I_b ln\frac{D_3}{D_1}\right) + I_a\frac{\mu}{2} \tag{143}$$

since, as S approaches infinity, all the values of S cancel out.

The above equation may be written

$$\phi_a = 2\left(I_a ln\frac{1}{r} + I_b ln\frac{1}{D_1} - (I_a + I_b) ln\frac{1}{D_3}\right) + I_a\frac{\mu}{2}$$

or, since

$$-(I_a + I_b) = I_c$$

$$\phi_a = 2\left(I_a ln\frac{1}{r} + I_b ln\frac{1}{D_1} + I_c ln\frac{1}{D_3}\right) + I_a\frac{\mu}{2}. \tag{144}$$

Similarly,

$$\phi_b = 2\left(I_a ln\frac{1}{D_1} + I_b ln\frac{1}{r} + I_c ln\frac{1}{D_2}\right) + I_b\frac{\mu}{2}. \tag{145}$$

$$\phi_c = 2\left(I_a ln\frac{1}{D_3} + I_b ln\frac{1}{D_2} + I_c ln\frac{1}{r}\right) + I_c\frac{\mu}{2} \tag{146}$$

The above are the complex linkages per centimeter of wire in absolute units of current. To change these to practical units per mile of wire and $\log_{10}$, the quantities in the brackets must be multiplied by the conversion 370.56×10^{-6}, while the last term of each equation is multiplied by the factor 160.94×10^{-6}. Thus, in henries per mile and amperes,

$$\phi_a = 741.13 \times 10^{-6}\left(I_a \log_{10}\frac{1}{r} + I_b \log_{10}\frac{1}{D_1} + I_c \log_{10}\frac{1}{D_3}\right) + 80.47 \times 10^{-6}\,\mu I_a \tag{147}$$

$$\phi_b = 741.13 \times 10^{-6}\left(I_a \log_{10}\frac{1}{D_1} + I_b \log_{10}\frac{1}{r} + I_c \log_{10}\frac{1}{D_2}\right) + 80.47 \times 10^{-6}\,\mu I_b \tag{148}$$

$$\phi_c = 741\ 13 \times 10^{-6}\left(I_a \log_{10}\frac{1}{D_3} + I_b \log_{10}\frac{1}{D_2} + I_c \log_{10}\frac{1}{r}\right) + 80.47 \times 10^{-6}\mu I_c. \tag{149}$$

Given the three complex potential differences at the supply end of the above three-phase line, together with the three expres-

sions for the complex currents, I_a, I_b and I_c, the drop in each wire can be calculated, and, by subtracting these from the supply voltages, the receiver voltages may be found.

Let the three potential differences to neutral at the supply end be

$$\left.\begin{aligned} E_a &= E(1 + j0) \\ E_b &= -\frac{E}{2}(1 + j\sqrt{3}) \\ E_c &= \frac{E}{2}(-1 + j\sqrt{3}) \end{aligned}\right\} \tag{150}$$

and assume the three complex currents in the wires to be

$$\left.\begin{aligned} I_a &= I_{a_1}(\cos\theta - j\sin\theta) \\ I_b &= I_{b_1}\left[\cos\left(\theta + \frac{2\pi}{3}\right) - j\sin\left(\theta + \frac{2\pi}{3}\right)\right] \\ I_c &= I_{c_1}\left[\cos\left(\theta + \frac{4\pi}{3}\right) - j\sin\left(\theta + \frac{4\pi}{3}\right)\right] \end{aligned}\right\} \tag{151}$$

where I_{a_1}, I_{b_1} and I_{c_1}, are the amplitudes of the corresponding complex currents I_a, I_b and I_c. The impedance drop in conductor a is

$$\begin{aligned} Z_a I_a &= (R + j\omega L_a)I_a \\ &= RI_a + j\omega\phi_a. \end{aligned} \tag{152}$$

By substituting in Eq. (152) the value of I_a from Eq. (151), and the value of ϕ_a from Eq. (147), the drop may be calculated, since the currents I_a, I_b and I_c are all assumed to be known and defined by Eq. (151). The impedance drops in the remaining two conductors may be found in a similar manner.

Conductor Arrangements and Transpositions.—The conductors of three-phase transmission lines are supported on poles or towers in various arrangements, depending upon the type of supports used, the voltage of the line, the number of circuits per tower line, etc. In general, however, the three conductors are supported either at the three corners of a triangle, or they all lie in a single plane, usually a horizontal plane. When the triangular arrangement is used, and the sides of the triangle are all equal, the three loops of the line are balanced. That is, the inductive and condensive reactances of all loops are the same. This is the only arrangement of untransposed conductors for which such balance exists. While it is general practice to transpose the conductors of power lines in order to minimize possible inductive inter-

ference with adjacent lines, particularly with communication circuits, it is desirable to transpose three-phase lines having unsymmetrical conductor arrangements in order to secure like performance in the three phases.

Balanced phases are secured in three-phase lines by so arranging the conductors that each of the three conductors occupies each of the three possible positions throughout a total of one-third of the length of the line. Two complete transpositions of the line, dividing it into three equal sections, is the minimum number which will accomplish the desired result.

Equilateral Arrangement. Value of L. In this type of arrangement the conductors form the three edges of an equiangular prism, as illustrated in Fig. 29. The spacings are equal, and the flux through the loop formed by any two conductors is independent of the magnetizing force of the current in the third conductor. Thus for loop AB, since $D_1 = D_2 = D_3 = D$,

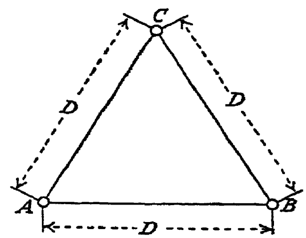


FIG 29.—Equilateral spacing for a three-phase line.

$$\phi_{ab} = 2ln\frac{D}{r}(I_a - I_b) \text{ vector linkages.} \tag{153}$$

From symmetry it is apparent that the partial self-inductances per unit length are equal for all conductors; that is, ${}_1L_a = {}_1L_b = {}_1L_c = L_1$. Also, the resultant vector linkages of a loop are equal to the vector differences of the linkages contributed by the individual conductors. Thus,

$$\phi_{ab} = L_1(I_a - I_b) \text{ vector linkages} \tag{154}$$

whence, from Eq. (153),

$$L_1 = 2ln\frac{D}{r} \text{ abhenries per centimeter of conductor.} \tag{155}$$

If we add the linkages per ampere within the conductor to the result expressed by Eq. (155), substitute $\log_{10}$ and write the result in terms of henries per mile of one conductor, the total inductance per mile of one conductor is found. It is

$$L = (741.13 \log_{10} \frac{D}{r} + 80.47\mu)10^{-6} \text{ henries per mile of one conductor.} \tag{156}$$

This equation is identical with Eq. (133). That is, the inductance of one conductor of a three-phase equilateral line is the same as that for a single-phase line having like conductors and spacings.

Unsymmetrically Arranged, Transposed, Three-phase Lines. Value of L.—Consider a three-phase line arranged as illustrated in Fig. 30, in which the sides D_1, D_2 and D_3 of the triangle may have any possible relative values. The values of D_1, D_2 and D_3 are large as compared with r, so that $(D - r) \div r$ is approximately equal to $D \div r$. Assume the line to be transposed so that the three phases are balanced. In short lines, so far as the power circuit alone is concerned, regardless of how many complete transpositions the line may actually have (so long as the number

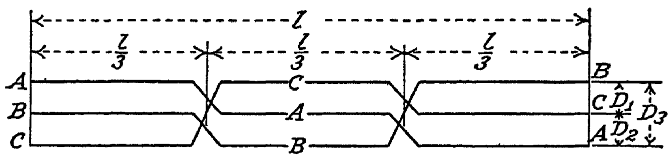


FIG. 30.—Unsymmetrically arranged three-phase line with transpositions

is not zero), the line is approximately electrically equivalent to a like line having only one complete spiral, as illustrated in the figure. For very long lines this is not strictly true, since both the line voltages between conductors and the currents in them may differ considerably between the two ends of a line.

Since the line is transposed, each of the conductors in turn occupies each of the three possible positions in the circuit. The numerical value of the flux-linkages is the same for all conductors, and all conductors have the same coefficient of self-inductance. It will, therefore, be necessary to find the average vector flux linkages for one loop only. This is done by computing the linkages per centimeter of loop for each of the three conductor arrangements in the transposed line and averaging them. Thus (neglecting the linkages within the conductors), for the loop ab, the linkages in each of the positions are

$$ {}_1\phi_{ab} = 2\left(I_a ln\frac{D_1}{r} - I_b ln\frac{D_1}{r} + I_c ln\frac{D_2}{D_3}\right) $$

$$ {}_2\phi_{ab} = 2\left(I_a ln\frac{D_2}{r} - I_b ln\frac{D_2}{r} + I_c ln\frac{D_3}{D_1}\right) $$

$$ {}_3\phi_{ab} = 2\left(I_a ln\frac{D_3}{r} - I_b ln\frac{D_3}{r} + I_c ln\frac{D_1}{D_2}\right) $$

and the average flux linkages, per centimeter of loop of the transposed line is one-third the sum of the above partial linkages; or

$$\phi_{ab} = \frac{2}{3}(I_a - I_b)ln\frac{D_1D_2D_3}{r^3}. \tag{157}$$

Since the line is transposed, and therefore electrically balanced, the partial inductances of the three like conductors are equal. Representing these as ${}_1L_a$, ${}_1L_b$ and ${}_1L_c$, we have

$${}_1L_a = {}_1L_b = {}_1L_c = L_1.$$

Furthermore, the partial inductance of the loop is such that

$${}_1L_a(I_a - I_b) = \phi_{ab} = \frac{2}{3}(I_a - I_b)ln\frac{D_1D_2D_3}{r^3}$$

or

$$L_1 = {}_1L_a = \frac{2}{3}ln\frac{D_1D_2D_3}{r^3} \text{ abhenries per centimeter.} \tag{158}$$

Adding to this the coefficient due to the linkages lying within the conductor, and reducing to practical units and $\log_{10}$,

$$L = \left(741.13 \log_{10} \frac{\sqrt[3]{D_1D_2D_3}}{r} + 80.47\right) \times 10^{-6}$$
$$\text{henries per mile of conductor.} \tag{159}$$

Equivalent Spacing.—If we assume an untransposed loop of an equilaterally arranged line of spacing D', such that its inductance per unit length of conductor is equal to that found in Eq. (159) for the unequally spaced, transposed line, the spacing D' may be called the *equivalent equilateral spacing* of the unequally spaced line. This is equivalent to equating the right-hand members of Eqs. (155) and (158). That is,

$$L_1 = 2ln\frac{D'}{r} = \frac{2}{3}ln\frac{D_1D_2D_3}{r^3}$$

and

$$D' = \sqrt[3]{D_1D_2D_3}. \tag{160}$$

Thus, the partial inductance L_1 per mile of conductor of any transposed, irregularly spaced line (or of an equilaterally spaced line whether transposed or not), may be calculated from Eq. (156) if the equivalent spacing given by Eq. (160) is used.

For the so-called flat spacing arrangement, the three conductors lie in a single plane, usually a horizontal or a vertical plane for high-voltage lines. Furthermore, the distance between conductors is usually such that $D_1 = D_2 = \frac{D_3}{2}$. For this case the equivalent separation is

$$D' = D\sqrt[3]{2}$$
$$= 1.26D. \tag{161}$$

The Inductance of Untransposed, Double-circuit, Three-phase Lines.—Consider two untransposed, three-phase circuits as in Fig. 31, in which opposite conductors constitute the two conductors of a phase. The conductors of each pair are in parallel between generating and receiving station busses, and hence have the same impedance drop. Furthermore, the vector sum of the six currents, I_a, $I_{a'}$, I_b, $I_{b'}$, I_c, and $I_{c'}$ must equal zero. These are the basic conditions from which inductance of each wire may be calculated. If the arrangement is a symmetrical one, as in this instance, the currents are equal in pairs, whence

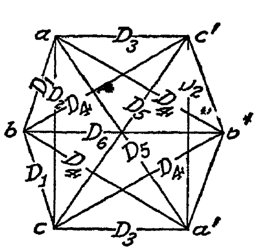


FIG 31 —The double-circuit three-phase line.

$$I_a = I_{a'}$$
$$I_b = I_{b'}$$
$$I_c = I_{c'}.$$

In this case there are only three unknown currents instead of six, and the problem is accordingly considerably simplified.

It is apparent that in any unsymmetrical arrangement the inductance will have a different value for each conductor, while, with a symmetrical arrangement like that illustrated here, the numerical inductances of the outside wires a and c are equal for balanced currents; but the latter will not equal the inductance of b, except in the special case when $D_1 = D_3$.

The method of solving for L will be illustrated for the two-circuit, three-phase line of Fig. 31, in which half the current of a phase flows in each of the corresponding phase cables, as stated above. Let r be the radius of each of the equal cables, let S be a very large but finite distance, and let both r and S be measured in the same units as the distances D_1, D_2, etc. Assume the vector currents to be given by the equations

$$\left.\begin{aligned} I_a &= I_{a'} = \frac{I_0}{2}(1 + j0) \\ I_b &= I_{b'} = \frac{-I_0}{4}(1 + j\sqrt{3}) \\ I_c &= I_{c'} = \frac{I_0}{4}(-1 + j\sqrt{3}) \end{aligned}\right\} \tag{162}$$

The total vector flux linkages, per centimeter for the three conductors, are

$$\left.\begin{aligned} \phi_a &= 2\Big(I_a ln\frac{S}{r} + I_b ln\frac{S}{D_1} + I_c ln\frac{S}{D_2} \\ &\qquad + I_{a'} ln\frac{S}{D_5} + I_{b'} ln\frac{S}{D_4} + I_{c'} ln\frac{S}{D_3}\Big) + I_a\frac{\mu}{2} \\ \phi_b &= 2\Big(I_a ln\frac{S}{D_1} + I_b ln\frac{S}{r} + I_c ln\frac{S}{D_1} \\ &\qquad + I_{a'} ln\frac{S}{D_4} + I_{b'} ln\frac{S}{D_6} + I_{c'} ln\frac{S}{D_4}\Big) \\ \phi_c &= 2\Big(I_a ln\frac{S}{D_2} + I_b ln\frac{S}{D_1} + I_c ln\frac{S}{r} \\ &\qquad + I_{a'} ln\frac{S}{D_3} + I_{b'} ln\frac{S}{D_4} + I_{c'} ln\frac{S}{D_5}\Big) + I_c\frac{\mu}{2} \end{aligned}\right\} \tag{163}$$

Substituting in these equations the values of I_a, $I_{a'}$, I_b, $I_{b'}$, I_c and $I_{c'}$, in terms of I_0, from Eq. (162), we get

$$\phi_a = I_0(1 + j0)\left(\frac{1}{2}ln\frac{D_1D_2D_3D_4}{r^2D_5^2} + \frac{\mu}{4} + j\frac{\sqrt{3}}{2}ln\frac{D_1D_4}{D_2D_3}\right) \tag{164}$$

$$\phi_b = -I_0\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(ln\frac{D_1D_4}{rD_6} + \frac{\mu}{4}\right) \tag{165}$$

$$\phi_c = I_0\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}ln\frac{D_1D_2D_3D_4}{r^2D_5^2} + \frac{\mu}{4} - j\frac{\sqrt{3}}{2}ln\frac{D_1D_4}{D_2D_3}\right). \tag{166}$$

Since, in general, $LI = \phi$, it is apparent that in absolute units the inductances are obtained by dividing the vector flux linking each conductor by the vector current in the conductor. Performing this operation and reducing to practical units yields the inductances given below:

$$L_a = 10^{-5}\left(18.53 \log_{10}\frac{D_1D_2D_3D_4}{r^2D_5^2} + 40.2\mu + j32.09 \log_{10}\frac{D_1D_4}{D_2D_3}\right) \text{ henries per mile} \tag{167}$$

$$L_b = 10^{-5}\left(37.06 \log_{10}\frac{D_1D_4}{rD_6} + 40.2\mu\right) \text{ henries per mile} \tag{168}$$

$$L_c = 10^{-5}\left(18.53 \log_{10}\frac{D_1D_2D_3D_4}{r^2D_5^2} + 40.2\mu - j32.09 \log_{10}\frac{D_1D_4}{D_2D_3}\right) \text{ henries per mile.} \tag{169}$$

Inductance of Transposed, Double-circuit, Three-phase Line. It will be of interest to consider the effect on the inductance of a transposed, three-phase line of the close proximity, of a second similar transposed line, operated in parallel with the first.

Let the lines be spaced as in Fig. 31 and assume each of the lines to have sufficient transpositions to balance the phases. It is assumed further that the paralleled pairs consist of the conductors a and a', b and b' and c and c', respectively, and that the transpositions are so made that the conductors of a pair such as a and a', etc., are always opposite each other in the figure. The inductance per centimeter of conductor will then be the same for all conductors.

There are six currents, each presumably contributing something to the linkages encircling a given conductor, or linking a given loop. Because of the symmetrical construction, however, and since the two lines are operated in parallel, the vector currents are equal in pairs, whence

$$I_a = I_{a'}$$
$$I_b = I_{b'}$$
$$I_c = I_{c'}$$

so that, in reality, only three currents remain to be considered.

Writing the partial flux-linkages per centimeter for the loop ab in each of its three positions, we find that

$$\left.\begin{aligned}
{}_1\phi_{ab} &= 2\left(I_a ln\frac{D_1}{r} - I_b ln\frac{D_1}{r} - I_c ln\frac{D_2}{D_1} - I_a ln\frac{D_5}{D_4}\right.\\
&\qquad\qquad \left. - I_b ln\frac{D_4}{D_6} + I_c ln\frac{D_4}{D_3}\right)\\
{}_2\phi_{ab} &= 2\left(I_a ln\frac{D_1}{r} - I_b ln\frac{D_1}{r} + I_c ln\frac{D_2}{D_1} + I_a ln\frac{D_4}{D_6}\right.\\
&\qquad\qquad \left. + I_b ln\frac{D_5}{D_4} + I_c ln\frac{D_3}{D_4}\right)\\
{}_3\phi_{ab} &= 2\left(I_a ln\frac{D_2}{r} - I_b ln\frac{D_2}{r} + I_c ln\frac{D_1}{D_1} - I_a ln\frac{D_5}{D_3}\right.\\
&\qquad\qquad \left. + I_b ln\frac{D_5}{D_3} + I_c ln\frac{D_4}{D_4}\right)
\end{aligned}\right\} \quad (170)$$

Adding the above partial linkages and averaging by dividing by 3,

$$\phi_{ab} = \frac{2}{3}\left[(I_a - I_b) ln\frac{D_1^2 D_4^2 D_2 D_3}{r^3 D_5^2 D_6}\right]. \quad (171)$$

Since, for a balanced line,

$$\phi_{ab} = L_1(I_a - I_b)$$

$$L_1 = 2ln\frac{1}{r}\sqrt[3]{\frac{D_4^2 D_1^2 D_2 D_3}{D_5^2 D_6}} \text{ abhenries per centimeter of wire.} \quad (172)$$

Adding the coefficient due to the linkages within the conductor and converting to practical units,

$$L = 10^{-6}\left(741\ 13 \log_{10}\frac{1}{r}\sqrt[3]{\frac{D_4^2 D_1^2 D_2 D_3}{D_5^2 D_6}} + 80.45\mu\right) \text{ henries per mile of one wire.} \quad (173)$$

Example.—The 250,000-cir. mil conductors of a double-circuit, 110-kv. three-phase line are spaced much as in Fig. 31. The vertical separation is 10 ft, the horizontal offset of the middle conductor is 5 ft, and the horizontal spacing D is 20 ft. The lines are transposed as described above. Required to find, (*a*) the reactance at 60 cycles per mile of conductor, assuming that one of the circuits is removed; (*b*) the reactance at 60 cycles per mile of conductor, assuming the presence of both lines with the spacings given; (*c*) the per cent change in reactance based on the reactance of a single line.

Solution.—Using the spacings given, the following values are found

$$\begin{aligned} D_1 &= \sqrt{10^2 + 5^2} = 11\ 18 \text{ ft.} & D_4 &= \sqrt{10^2 + 25^2} = 26\ 93 \text{ ft.} \\ D_2 &= \phantom{\sqrt{10^2+5^2}} = 20 \text{ ft.} & D_5 &= \sqrt{20^2 + 20^2} = 28\ 25 \text{ ft.} \\ D_3 & \phantom{\sqrt{10^2+5^2}} = 20 \text{ ft.} & D_6 &= \phantom{\sqrt{20^2+20^2}} = 30\ 0 \text{ ft.} \end{aligned}$$

$$r = \frac{0\ 576}{2 \times 12} = 0.0240 \text{ ft.}$$

For the single line,

$$\frac{D'}{r} = \frac{\sqrt[3]{11\ 18 \times 11\ 18 \times 20}}{0.0240} = 565.8$$

$$\log_{10} 565.8 = 2\ 753$$

$$\begin{aligned} x = \omega L &= 377(741\ 13 \times 2.753 + 80.45)10^{-6} \\ &= 0.7995 \text{ ohm per mile.} \end{aligned}$$

For the double-circuit line,

$$\begin{aligned} \frac{D'}{r} &= \frac{1}{r}\sqrt[3]{\frac{11\ 18^2 \times 26.93^2 \times 20 \times 20}{28.28^2 \times 30}} \\ &= 476.3 \\ \log_{10} 476.3 &= 2.6778 \\ x = \omega L &= 377(741.13 \times 2.6778 + 80.45)10^{-6} \\ &= 0.778 \text{ ohm per mile.} \end{aligned}$$

Due to the presence of a second, similar circuit, the reactance of each circuit is decreased by the percentage,

$$\frac{100(0.799 - 0.778)}{0.799} = 2.63 \text{ per cent.}$$

Influence of Stranding and Spiraling on the Value of L.—Conductors of the larger sizes are made up of varying numbers of smaller, round wires twisted about a central conductor or group of conductors to form a stranded cable, as illustrated in Fig. 32, for a seven-strand cable. Owing to the somewhat larger space occupied by such a conductor, as compared with a solid, circular conductor of equal area, its inductance is somewhat smaller than for the equivalent solid rod. (The equivalent solid rod is a solid, round conductor having the same conducting area as the stranded cable.)

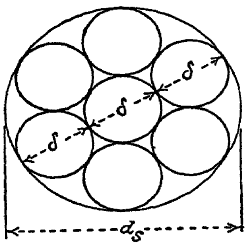


FIG 32.—Section of a seven-strand cable.

The exact value for a given cable depends upon its size, the number of strands and the pitch of the spiral used. Spiraling tends to increase the inductance of the cable, since it produces a solenoidal effect. Since the pitch of the spiral is large, however, this effect is quite negligible in transmission-line cables.

Equations[1] have been developed for calculating the inductances of transmission-line cables for cables of standard stranding, and for the various commercial sizes. There is little to be gained, however, by attempting too great a refinement in the calculation of inductance. For variations from the assumed values, in the spacings and in the actual lengths of cables, together with the influence of the tower in somewhat increasing the inductance where steel towers are used, may leave a larger margin of uncertainty than that introduced by the use of the approximate equation.

It is customary, in dealing with stranded cables for aerial power transmission work, to calculate their inductances by the use of Eq. (156), in which, however, it is preferable that the value of r is taken as the radius of the equivalent solid rod. This substitution yields values of inductance which are somewhat too high. The error is therefore on the safe side.

For example, a seven-strand cable is made up of seven round wires of diameter δ each, the seven being twisted together to form a cable of outside diameter $d_s = 3\delta$. Neglecting the effect of spiraling, the cross-sectional area of the cable in circular mils is

$$\text{cir. mil} = 7\delta^2 = 7\left(\frac{d_s}{3}\right)^2.$$

[1] DWIGHT, H. B., "Transmission Line Formulas."

If d be the diameter of the equivalent solid rod, it follows that

$$7\left(\frac{d_s}{3}\right)^2 = d^2$$

or

$$d_s = 3\sqrt{\frac{d^2}{7}}$$

and

$$d_s = 1.1338d.$$

The value of r to substitute in Eq. (141) is

$$r = \frac{d_s}{2 \times 1.1338}.$$

Table 6 gives the values of the overall diameters in terms of circular mil areas, and the corresponding ratios of $d_s \div d$ for cables of standard stranding up to cables of 127 strands.

TABLE 6

Number of strands	d_s	$d_s \div d$
1	$\sqrt{\text{cir. mil}}$	1
7	$3\sqrt{\frac{\text{cir. mil}}{7}}$	$\frac{3}{\sqrt{7}} = 1.1338$
19	$5\sqrt{\frac{\text{cir. mil}}{19}}$	$\frac{5}{\sqrt{19}} = 1.1471$
37	$7\sqrt{\frac{\text{cir. mil}}{37}}$	$\frac{7}{\sqrt{37}} = 1.1508$
61	$9\sqrt{\frac{\text{cir. mil}}{61}}$	$\frac{9}{\sqrt{61}} = 1.1523$
91	$11\sqrt{\frac{\text{cir. mil}}{91}}$	$\frac{11}{\sqrt{91}} = 1.1531$
127	$13\sqrt{\frac{\text{cir. mil}}{127}}$	$\frac{13}{\sqrt{127}} = 1.1536$

PROBLEMS

1. The two round conductors of a parallel-sided loop are each 0.60 in. in diameter. The distance between centers of conductors is 12 ft. The current in the conductors is 225 amp. In a sectional view taken normal to the plane of the loop, draw 5 lines of equal magnetic potentials, so spacing the lines that the magnetic potential drop between each 2 adjacent lines shall be one-eighth of the total magnetomotive force of the loop. Let the

straight line joining the two conductors be one of the lines, as well as the axis of symmetry, of the completed figure.

2. For the circuit of Problem 1, in the sectional view, draw magnetic lines of force so spaced that each 2 adjacent lines shall include between them one-tenth of the total flux through the loop Let the straight line drawn normal to the plane of the loop and midway between conductors, be one of the lines, as well as the axis of symmetry of the figure. How much flux passes between adjacent lines per mile of circuit when the current is 225 amp.?

3. An untransposed, split-conductor, single-phase, 60-cycle line, 2,000 ft. long, is built of four 00 copper conductors, each having a resistance of 0 46 ohm per mile The conductors lie in a horizontal plane, and reading from left to right, their order is a, b, a', b' The conductors ab form the outgoing pair, while $a'b'$ are the corresponding return conductors. The distances of separation ab and $a'b'$ are each 2 ft., while ba' is 4 ft. When the impressed voltage is 2,300 volts, and the load current is 220 amp., what is the potential difference $E_{ab'}$ at the receiver end of the line?

4 An untransposed, three-phase, 60-cycle line has three 0000 conductors of diameters 0.533 in spaced 6 ft apart in a horizontal plane The resistance of the conductors at working temperatures is 0.28 ohm per mile. The currents in the conductors are unbalanced and equal to

$$\left.\begin{aligned} I_a &= 156 - j\ 90 \\ I_b &= -111 - j\ 108 \\ I_c &= -\ 45 + j\ 198 \end{aligned}\right\} \text{vector amp.}$$

The supply voltages to neutral are balanced, and have the following complex values.

$$\left.\begin{aligned} E_a &= 3,800(1 + j0) \\ E_b &= -1,900(1 + j\sqrt{3}) \\ E_c &= 1,900(-1 + j\sqrt{3}) \end{aligned}\right\} \text{vector volts.}$$

Find the complex expressions for the line-to-line voltages E_{ab}, E_{bc} and E_{ca} at the receiver end.

5. What is the 60-cycle, inductive reactance per mile of a conductor for the transposed, double-circuit line in Problem 5, Chap. IV?

CHAPTER IV

THE DIELECTRIC CIRCUIT AND CAPACITANCE

Any space in which a charge of electricity experiences a force, may be called a dielectric field. Dielectric fields are represented by dielectric *lines of force* in much the same way as magnetic fields are represented by magnetic lines of force. At any point in a dielectric field the direction and sense of the dielectric lines of force indicate the direction and sense of the resultant force on unit positive quantity of electricity at that point. The magnitude of the force is indicated by the number of lines crossing per unit of area taken normal to their path, that is, by the dielectric flux density.

Concepts, Definitions and Units.—Unless otherwise stated, the units used in the definitions and discussions which follow are the *electrostatic units*.

Measurements of dielectric field intensities are based on the concept of unit point charge of electricity. The charge is assumed to be collected at a point, and to be of such amount that, in air and when separated from an equal charge of like sign, it will experience a force of 1 dyne. Such a charge is called a *unit point charge*.

The *electrical potential of a point* is the work done on a unit positive point charge in bringing it from the edge of the dielectric field, where the force on the charge is zero, up to the point in question.

The *field intensity* of a dielectric field is the force in dynes experienced by a unit positive point charge in the field. It is the electrical *potential gradient* at the point, measured along a line of force. This follows from the definition of potential given above. When measured in statvolts per centimeter the gradient is designated by the symbol K; when practical electromagnetic units are used it is measured in volts per centimeter, for which the symbol is G. G and K are related by the equation

$$G = \frac{4\pi v^2 K}{10^9} \tag{174}$$

where v is the velocity of propagation of the electric field in centimeters per second, and is equal approximately to the velocity of light.

That is,

$$v = 3 \times 10^{10} \text{ cm. per second.} \tag{175}$$

Since the direction of a dielectric line of force represents the direction of the resultant dielectric field at every point, the electrical *potential difference* between any two points in the field is the integral of the field intensity taken along a line of force. The field intensity is

$$K = \frac{de}{dl \cdot \cos \theta} \tag{176}$$

from which

$$e = \int_{(L)} K \cdot dl \cdot \cos \theta \tag{177}$$

where e = the potential difference between the two points defined by the limits of the integration.

dl = an elementary length of path.

θ = the angle between the direction of the path and the direction of a line of force at every point.

The unit of dielectric flux is a *dielectric line of force*. The total flux crossing a given area is represented by the symbol ψ, while the *flux density* is

$$D = \frac{\psi}{A} \tag{178}$$

where A is the area in square centimeters of path of the dielectric flux, taken normal to its direction. The unit of density is 1 line per square centimeter.

In a medium of air, a field intensity of 1 statvolt per centimeter sets up a flux density of 1 dielectric line per square centimeter. This is equivalent to saying that for the abstat system of units the dielectrical conductivity of air per cubic centimeter is unity. Materials other than air have various conductivities, and, in them, a field intensity of unity will accordingly produce various dielectric flux densities, depending upon the nature of the material The ratio

$$k = \frac{D}{K} \tag{179}$$

of flux density to field intensity, in a dielectric field, is called the *permittivity* of the material in which the field is established.

The *law of the dielectric circuit*, analogous to Ohm's law for the electric and magnetic circuits, is

$$E = \psi S$$

or

$$\psi = \frac{E}{S} \tag{180}$$

where E is the total potential difference required to set up the dielectric flux ψ over the length l of the dielectric circuit. The constant $S = l/kA$ is called the *elastance* of the dielectric circuit, analogous to the reluctance of a magnetic circuit. In practical electromagnetic units, the elastance is

$$S = \frac{4\pi v^2 l}{10^9 kA} \text{ darafs} \tag{181}$$

whence, using volts of potential difference and c.g.s. units of length and area, the law of the dielectric circuit, in practical electromagnetic units, becomes

$$E = \frac{4\pi v^2 \psi l}{10^9 kA} \text{ volts.} \tag{182}$$

When a difference of electrical potential is applied to a dielectric circuit, thereby producing a dielectric field, variations in potential difference are accompanied by corresponding changes in the dielectric flux. The varying flux is accompanied by a flow of electricity, that is, by an electric current, which flows in such a direction as to oppose the change in potential which causes it. The collapse of each unit of dielectric flux may be thought of as releasing a definite quantity of electricity. The current set up is proportional to the negative rate of change of flux and hence also to the negative rate of change of the potential difference. In abstat units, these relations are expressed by the equation,

$$i = -\frac{Cde}{dt} = -\frac{d\psi}{dt} \tag{183}$$

where C is the proportionality factor between the time rate of change of potential difference and the current. From Eq. (183) it follows that

$$CE = \psi = \frac{E}{S}$$

and

$$C = \frac{\psi}{E} = \frac{1}{S} \tag{184}$$

For any electrical circuit the constant C as defined by Eq. (184) is called the *capacitance* of the circuit. In stat units it is the number of dielectric flux lines set up per statvolt of potential difference impressed, called the *statfarad*.

The practical unit of capacitance is the *farad*. This unit is related to the stat unit as shown in Eq. (185).

$$\text{Farads} = \text{statfarads} \times \frac{10^9}{v^2} \tag{185}$$

By multiplying Eq. (183) through by $e \cdot dt$ and integrating, the stored energy of the dielectric field, when the potential difference is E, is found to be

$$E = \frac{E^2C}{2} \text{ joules} \tag{186}$$

where C is in farads and E is in volts.

Potential Gradient and Capacity.—In dealing with the circuits of transmission systems it is frequently necessary to compute both the potential gradients about the conductors or within the insulating materials which isolate them, and the capacitances between the various conductors or between one conductor and neutral. The potential gradient must be kept below the value at which the dielectric medium breaks down, and the capacitance must be known in order that the operating characteristics of the circuit may be predetermined. In the simplest cases these calculations are quite readily made or at least may be closely approximated. The general method of making them is much the same in all cases, as will be apparent from the illustrations which follow.

The Parallel-plate Condenser.—A condenser consisting of two equal parallel plates separated by a homogeneous dielectric

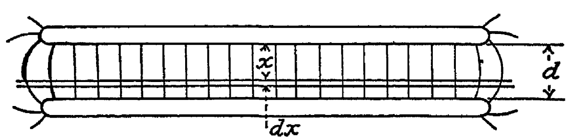


FIG. 33.—The parallel-plate condenser.

material is perhaps the simplest form of a capacitance. Such a condenser is illustrated in Fig. 33.

Let d = the distance between plates in centimeters.
A = the number of square centimeters area per plate.
E = the potential difference between the equipotential plates.
ψ = the total number of dielectric lines passing between plates.

Let it be assumed that, compared with the area of the plates, the distance of separation is small, so that the distortion of the field around the edges of the plates is negligible, and the entire field may be considered uniform. The elastance of an elementary solid, with lateral faces parallel to the plates of the condenser, of width dx and of area equal to that of one of the plates is

$$\begin{aligned} dS &= \frac{4\pi v^2 dx}{10^9 kA} \\ &= 4\pi\lambda\frac{dx}{A} \end{aligned} \tag{187}$$

where k = the permittivity of the dielectric material.

$\lambda = \dfrac{v^2}{10^9 k}$ (used to simplify the notation).

The drop of potential between parallel faces of the elementary solid is

$$\begin{aligned} de &= \psi dS \\ &= 4\pi\lambda\psi\frac{dx}{A}. \end{aligned} \tag{188}$$

The potential gradient in the dielectric is

$$\begin{aligned} G &= \frac{de}{dx} \\ &= \frac{4\pi\lambda\psi}{A} \text{ volts per centimeter.} \end{aligned} \tag{189}$$

The gradient is seen to be independent of x and therefore constant throughout the dielectric.

The total flux in the dielectric is obtained from Eq. (188) by equating the integral of the right-hand member, taken between the limits of 0 and d of the variable, to the potential difference between plates, whence

$$E = \frac{4\pi\lambda\psi d}{A} \text{ volts}$$

and

$$\psi = \frac{AE}{4\pi\lambda d} \text{ lines.} \tag{190}$$

By Eq. (184) the capacitance of the condenser is

$$C = \frac{\psi}{E}$$

$$= \frac{A}{4\pi\lambda d} \text{ farad} \tag{191}$$

$$= \frac{8.842kA \times 10^{-3}}{d} \text{ mf.} \tag{192}$$

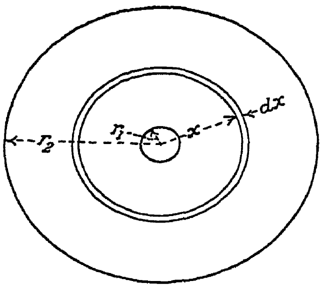


FIG 34.—The lead-covered cable.

Concentric Cylinders.—The single-conductor, lead-covered cable illustrates this case. It consists of a single conductor of circular cross-section covered with a certain thickness of homogeneous, insulating material, the whole, in turn, being enclosed in a lead sheath, as in Fig. 34. Let the sheath of the cable be grounded, and let the potential difference of the conductor to neutral be E_0. It is assumed that the drop of potential in the conductor is negligible, and hence both the conductor and the sheath are equipotential surfaces.

Let r_1 = radius of conductor in centimeters.
r_2 = inside radius of the sheath.
ψ = the total number of dielectric lines per centimeter of conductor length, passing between core and sheath.

The elastance of any elementary, concentric cylinder of the dielectric, of unit length, of radius x and of thickness of wall dx is

$$dS = \frac{4\pi v^2 dx}{10^9 \cdot 2\pi k x}$$

$$= \frac{2\lambda dx}{x}.$$

The potential drop across the walls of the elementary cylinder is

$$de = \psi dS$$

$$= \frac{2\lambda\psi dx}{x}$$

whence the total potential difference between the core and sheath is

$$E_0 = 2\lambda\psi \int_{r_1}^{r_2} \frac{dx}{x}$$
$$= 2\lambda\psi ln \frac{r_2}{r_1} \text{ volts.} \tag{193}$$

Accordingly, the flux passing out from the core, per unit length of conductor, is

$$\psi = \frac{E_0}{2\lambda ln \frac{r_2}{r_1}} \tag{194}$$

and the corresponding capacitance is

$$C_0 = \frac{\psi}{E_0}$$
$$= \frac{1}{2\lambda ln \frac{r_2}{r_1}} \text{ farads per centimeter} \tag{195}$$
$$= \frac{0\ 03883k}{\log_{10} \frac{r_2}{r_1}} \text{ mf per mile.} \tag{196}$$

The potential gradient within the dielectric at a distance of x cm. from the center of the core is

$$G = \frac{de}{dx}$$
$$= \frac{2\lambda\psi}{x} \text{ volts per centimeter} \tag{197}$$

Substituting the value of ψ from Eq. (194) in Eq. (197) transforms the latter to

$$G = \frac{E_0}{xln \frac{r_2}{r_1}} \text{ volts per centimeter.}$$
$$= \frac{0.4343E_0}{x \log_{10} \frac{r_2}{r_1}} \text{ volts per centimeter.} \tag{198}$$

Eq. (198) shows that, for this case, the gradient is maximum at the surface of the conductor and varies inversely as the distance from the center of the core.

The Dielectric Field about a Long, Straight Cylinder.—At any point distant r cm. from a long, straight cylinder suspended in a medium of constant permittivity k, the field intensity is given by the equation,

$$K = \frac{2Q}{kr}$$

$$= \frac{\psi}{2\pi kr} \text{ abstat units per centimeter} \qquad (199)$$

where $\psi = 4\pi Q$, the number of dielectric lines leaving the conductor per centimeter of its length.

This conclusion results from a proof very similar to that given in the preceding chapter for the magnetic field intensity near a long straight wire.

Equation (199) is reduced to practical, electromagnetic units by introducing the conversion factor $\lambda = \frac{v^2}{10^9 k}$, whence, by Eq. (174),

$$G = \frac{4\pi v^2 k'}{10^9}$$

$$= \frac{2\lambda\psi}{r} \text{ volts per centimeter.} \qquad (200)$$

Equation (200) shows that all points distant r cm. from the center of the cylinder have the same value of potential gradient. At all points about the conductor the gradient is radially directed. Since dielectric lines of force are everywhere drawn in the direction of the resultant gradient, they, too, are radial lines. Normal to these are the lines of equal potentials which, by Equation (199), are circles concentric with the conductor axis. In Fig. 35 these lines are shown, the equipotential circles as broken, and the lines of force as full lines. It should be noted that the lines of constant potential are also magnetic lines of force, while the dielectric lines of force are

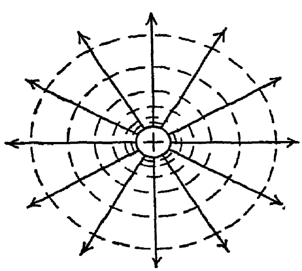


FIG. 35.—The dielectric field about an isolated straight wire.

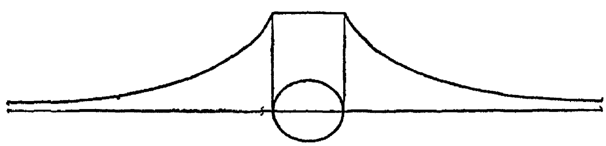


FIG. 36.—The potential gradient about a straight round wire.

the lines of constant magnetic potentials (compare Fig. 35, with Fig. 17).

Figure 36 shows how the potential gradient varies in the region on and outside of the isolated cylindrical conductor.

The Dielectric Field about Two Equal, Parallel, Cylindrical Conductors.—Let the equal, parallel, cylindrical conductors A and B of Fig. 37 be suspended in a medium of constant permittivity k. Assume that the distance of separation D of the conductors is great as compared with the radius of the conductor. Then the charges on the conductors, due to their potential difference, will be approximately uniformly distributed over the conductor surfaces. Let the equal charges on A and B, per unit length of conductor, be $+Q$ and $-Q$ units, respectively. These give rise to the corresponding, outwardly directed, equal dielectric fluxes $+\psi$ and $-\psi$ lines per centimeter. Let P be a point which moves along a line of the resultant dielectric field due to the potentials of A and B, and let the fixed point F, lying on the normal bisector of AB, be a point on the locus of P. The dielectric fluxes per centimeter length of conductor passing *outwardly* through the loop PF, are

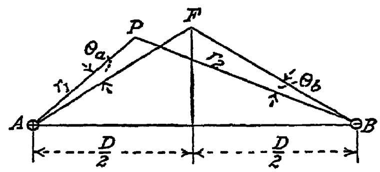


FIG. 37.—The dielectric lines of force between two equal, parallel, cylindrical conductors of a return loop are circles.

$$\left.\begin{aligned} \psi_a &= \frac{\psi\theta_a}{2\pi} \\ \psi_b &= -\frac{\psi\theta_b}{2\pi} \end{aligned}\right\} \qquad (201)$$

Since P is assumed to move along a line of force of the resultant dielectric field, however, the potential gradient normal to its path must everywhere be zero, and the total outgoing flux through the loop PF is therefore likewise zero. Therefore,

$$\psi_a + \psi_b = 0$$

and

$$\theta_a = \theta_b. \qquad (202)$$

This condition defines a circle passing through the points A, P, F and B. Thus the dielectric lines of force of the resultant field are circles. They are also the circles of constant, magnetic potentials, as already shown in the previous chapter (see p. 43).

The Lines of Constant Potential about Two Equal, Parallel, Cylindrical Conductors.—The constant-potential lines for this case are also circles and are identical with the circular magnetic lines of force already discussed in the previous chapter. The proof is exactly similar to that already given for the magnetic lines of force. It will be given, however, for the sake of completing the parallelism between the two cases.

Let A and B of Fig. 38 be two equal, parallel, cylindrical conductors, and let their surfaces be equipotential surfaces subjected to a potential difference of $E = 2E_0$ volts. The conductors are assumed to be widely separated so that the charges on their surfaces are uniformly distributed, and, so far as the region beyond the conductor surfaces are concerned, they are therefore electrically equivalent to equal charges uniformly distributed along the length of conductor filaments coincident with the axis of the conductors. Again, as before, let the outwardly going equal, uniformly distributed, radial fluxes per centimeter length of the conductors be $+\psi$ and $-\psi$. Let the point P move along a line of constant electrical potential, and let F be the point on the line AB where the locus of P crosses it, distant m and n centimeters from A and B, respectively.

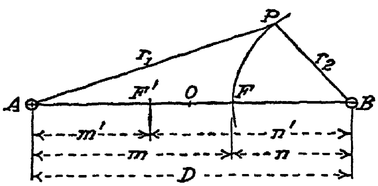


FIG. 38.—Lines of constant electrical potentials about two equal, parallel round wires of a return loop.

From the definition of potential difference and Eq. (200), the potential differences between F and P due to A and B each acting alone, are

$$E_a = 2\lambda\psi \int_m^{r_1} \frac{dr}{r}$$

and, similarly,

$$\left.\begin{aligned} E_a &= 2\lambda\psi ln\frac{r_1}{m} \\ E_b &= -2\lambda\psi ln\frac{r_2}{n} \end{aligned}\right\} \qquad (203)$$

Since, by assumption, P moves along a line of constant potential

$$E_a + E_b = 0$$

whence

$$\frac{r_1}{m} = \frac{r_2}{n}$$

and

$$\frac{r_1}{r_2} = \frac{m}{n} = \text{a constant.}$$

By the theorem of inverse points of a circle, this condition defines a circle passing through the points P and F. The equations of these circles are given in the previous chapter.

Figure 39 shows the two families of circles discussed in this and the previous article. The full lines represent the dielectric lines of force (also lines of constant, magnetic potentials), while the dotted lines are the lines of constant, electrical potential, and also the magnetic lines of force (Fig. 24).

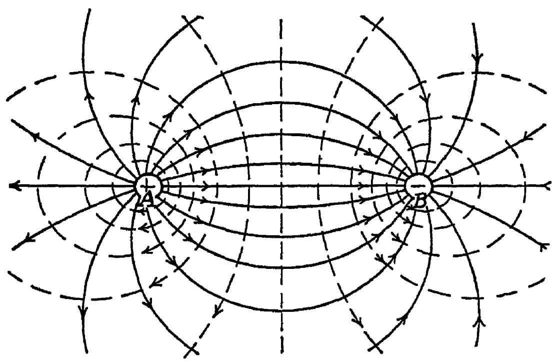


FIG. 39—The dielectric lines of force between the two parallel, cylindrical conductors of a return loop.

Calculation of Capacitance. General Method.—The general method of solving for the capacitance between two conductors, or between one conductor and a plane of zero potential, has already been illustrated in the solution of the problem for the concentric-core cable. The procedure was: First, to solve for the flux emanating from unit length of the conductor for any assumed potential difference; and secondly, to compute the capacitance by dividing the flux by the impressed potential difference. The same general procedure will be followed in finding the capacity for single-phase and three-phase lines.

The Capacitance of Two Parallel, Round Wires. Approximate Equation for Single-phase Line.—Let the two equal, parallel, cylindrical conductors A and B of Fig. 38 be suspended in a medium of constant permittivity k; let r be the radius of each of the cylinders, and assume that the separation D is large as compared with r. (This condition always prevails in high-

tension, aerial-transmission work.) As before, assume that the potential difference $E = 2E_0$ is impressed across the conductors, and is of such value that the resulting, uniformly distributed charge produced per unit length of the conductors is $+Q$ and $-Q$ for A and B, respectively. The corresponding outwardly radiating dielectric fluxes per unit length of conductor will then be $+\psi$ and $-\psi$ lines.

At the point P, the component potential gradients due to the separate potentials of the two conductors, are, by Eq (200),

$$\left.\begin{aligned} G_a &= \frac{2\lambda\psi}{r_1} \text{ directed outward along } r_1 \\ G_b &= -\frac{2\lambda\psi}{r_2} \text{ directed inward along } r_2 \end{aligned}\right\} \qquad (204)$$

The resultant gradient, made up of the vector sum of these two values, is at every point in the direction of the resultant line of force through P. The potential difference between any two points in a dielectric field is the line integral of the resultant potential gradient between the two points. Thus, in any plane normal to the conductors the potential drop between conductors is

$$E = \int_{(L)} G_0 \cos\theta \cdot dl$$

the integral to be taken from A to B over any path whatever,

where G_0 = the numerical value of the resultant gradient at P.
θ = the angle between the direction of the gradient and the path.
dl = the elementary length of path.

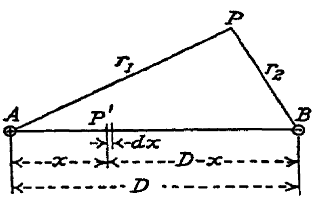


FIG. 40.—The capacitance of round, parallel wires.

This integral is most readily evaluated along the straight-line path from A to B, for along it the resultant gradient is at every point equal to the arithmetic sum of its two components, and its direction and sense are along the line AB from A towards B.

Thus, in Fig. 40, if P be moved to the position P', the component gradients become

$$G_a = +\frac{2\lambda\psi}{x}$$

$$G_b = -\frac{2\lambda\psi}{D - x}$$

and

$$G_0 = G_a - G_b$$

$$= 2\lambda\psi\left(\frac{1}{x} + \frac{1}{D - x}\right). \tag{205}$$

Therefore,

$$E = 2\lambda\psi \int_r^{D-r} \left[\frac{1}{x} + \frac{1}{D - x}\right] dx$$

$$= 4\lambda\psi \, ln \, \frac{D - r}{r} \text{ volts} \tag{206}$$

and

$$\psi = \frac{E}{4\lambda ln \dfrac{D - r}{r}} \tag{207}$$

The capacitance between wires is

$$C = \frac{\psi}{E}$$

$$= \frac{1}{4\lambda ln \dfrac{D - r}{r}} \text{ farads per centimeter.} \tag{208}$$

It is apparent that the potential of the point on AB midway between conductors is zero. Accordingly, if in deriving Eq. (206), the potential difference to neutral had been used in the left-hand member, and on the right the integration had been carried from r to $\frac{D}{2}$, the capacitance of one wire to neutral would result. It is just twice the value given in Eq. (208) Hence

$$C_0 = \frac{1}{2\lambda ln \dfrac{D - r}{r}} \text{ farads}$$

per centimeter (one wire to neutral). (209)

Since D is large compared with r, the substitution of D for $(D - r)$ is permissible. Making this substitution, writing in terms of $\log_{10}$, evaluating the constant, and giving k a value of unity for the permittivity of air, Eqs. (208) and (209) are converted to

Capacitance between two conductors,

$$C = \frac{3.68}{10^3 \log_{10} \frac{D}{r}} \text{ mf. per 1,000 ft.} \tag{210}$$

$$= \frac{0.01942}{\log_{10} \frac{D}{r}} \text{ mf. per mile} \tag{211}$$

Capacitance of one conductor to neutral

$$C_0 = \frac{7.36}{10^3 \log_{10} \frac{D}{r}} \text{ mf. per 1,000 ft.} \tag{212}$$

$$= \frac{3.883}{100 \log_{10} \frac{D}{r}} \text{ mf. per mile.} \tag{213}$$

While the above equations are not theoretically exact, the errors introduced by their use are entirely negligible for all practical transmission spacings. To illustrate, when the spacing of wires between centers is only five times the conductor diameter, these equations yield results 0.36 per cent too low. The error diminishes rapidly as the spacing increases.

The Exact Value of Capacitance between Two Parallel Cylinders.—When the parallel conductors are close together, more exact equations than those given above may be desired. Exact expressions covering the case of two parallel, round conductors are developed below.

It has already been shown that if the potential drop in the conductors themselves be neglected, the equipotential lines linking a parallel-sided loop of two circular conductors are a family of circles to which the equipotential surfaces of the conductors themselves belong. Since, however, no restriction is now placed on the distance of separation between the two opposite sides of the loop, it is no longer permissible to assume the filaments, along which the charges are uniformly distributed, to be coincident with the conductor axes. In general, it is apparent that in a return loop the filaments, which are the electrical equivalents of the actual, non-uniformly distributed charges on the conductors, will be somewhat closer together than are the conductor axes, as shown in Fig. 41. Let the separation between filaments be represented by $2s$ while the separation of the conductors is the slightly greater distance D.

From Fig. 38 and Eq. (203) it is apparent that the difference of potential between any point P, and any other point F on the line AB, due to the filaments at A and B, is

$$E_{FP} = 2\lambda\psi\left(ln\frac{r_1}{m'} - ln\frac{r_2}{n'}\right).$$

If the point F is halfway between A and B at O, then $m = n$, and the potential difference between O and P will be the electrical potential of P, since the potential of O is zero. Denoting the potential of P by E,

$$E = 2\lambda\psi ln\frac{r_1}{r_2}. \qquad (214)$$

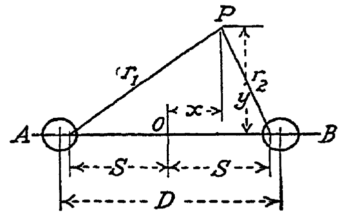


FIG. 41—The capacitance between two parallel, round wires that are close together

Using O as the origin of the cartesian coordinates, it is apparent that

$$\frac{r_1}{r_2} = \sqrt{\frac{(s+x)^2+y^2}{(s-x)^2+y^2}} \qquad (215)$$

which, substituted in (214), when simplified, yields

$$E = \lambda\psi ln\frac{(s+x)^2+y^2}{(s-x)^2+y^2} \qquad (216)$$

or

$$\epsilon^{\frac{E}{\lambda\psi}} = \frac{(s+x)^2+y^2}{(s-x)^2+y^2}. \qquad (217)$$

Equation (217), when expanded and simplified, results in

$$y^2 + x^2 + 2sx\left[\frac{1+\epsilon^{\frac{E}{\lambda\psi}}}{1-\epsilon^{\frac{E}{\lambda\psi}}}\right]^2 + s^2 = 0 \qquad (218)$$

or

$$y^2 + \left[x + s\left(\frac{1+\epsilon^{\frac{E}{\lambda\psi}}}{1-\epsilon^{\frac{E}{\lambda\psi}}}\right)\right]^2 = s^2\left[\left(\frac{1+\epsilon^{\frac{E}{\lambda\psi}}}{1-\epsilon^{\frac{E}{\lambda\psi}}}\right)^2 - 1\right]. \qquad (219)$$

This is the equation of a family of equipotential circles, the individuals of which are determined by the values assigned to E. The center of the circle has the coordinates

$$y = 0$$

$$x = -s\left(\frac{1+\epsilon^{\frac{E}{\lambda\psi}}}{1-\epsilon^{\frac{E}{\lambda\psi}}}\right) \qquad (220)$$

and the radius is

$$\rho = s\sqrt{\left(\frac{1 + \epsilon^{\frac{E}{\lambda\psi}}}{1 - \epsilon^{\frac{E}{\lambda\psi}}}\right)^2 - 1}$$

$$= \frac{2s\epsilon^{\frac{E}{2\lambda\psi}}}{1 - \epsilon^{\frac{E}{\lambda\psi}}}. \tag{221}$$

For $E = E_0$, the potential difference between the conductor A and the neutral plane through O, the equation yields the equipotential circle represented by a section normal to the circular conductor A. If the radius of the conductor is r, then, by Eq. (221),

$$r = \frac{2s\epsilon^{\frac{E_0}{2\lambda\psi}}}{1 - \epsilon^{\frac{E_0}{\lambda\psi}}}. \tag{222}$$

The center of the conductor is at

$$x = \frac{D}{2} = \frac{s\left(1 + \epsilon^{\frac{E_0}{\lambda\psi}}\right)}{1 - \epsilon^{\frac{E_0}{\lambda\psi}}} \tag{223}$$

whence

$$s = \frac{D}{2}\frac{\left(1 - \epsilon^{\frac{E_0}{\lambda\psi}}\right)}{\left(1 + \epsilon^{\frac{E_0}{\lambda\psi}}\right)}. \tag{224}$$

Substituting Eq. (224) in Eq. (222),

$$r = \frac{D}{\epsilon^{\frac{E_0}{2\lambda\psi}} + \epsilon^{-\frac{E_0}{2\lambda\psi}}}$$

$$= \frac{D}{2\cosh\frac{E_0}{2\lambda\psi}}. \tag{225}$$

That is,

$$\frac{D}{2r} = \cosh\frac{E_0}{2\lambda\psi}$$

or

$$\psi = \frac{E_0}{2\lambda\cosh^{-1}\frac{D}{2r}}. \tag{226}$$

Since

$$\psi = C_0 E_0$$

the capacitance of one conductor to neutral is

$$C_0 = \frac{1}{2\lambda \cosh^{-1}\dfrac{D}{2r}} \text{ farads per centimeter} \tag{227}$$

$$= \frac{8.948k}{100 \cosh^{-1}\dfrac{D}{2r}} \text{ mf. per mile.} \tag{228}$$

The capacitance between conductors is one half the value to neutral, or

$$C = \frac{1}{4\lambda \cosh^{-1}\dfrac{D}{2r}} \text{ farads per centimeter} \tag{229}$$

$$= \frac{4.474k}{100 \cosh^{-1}\dfrac{D}{2r}} \text{ mf per mile} \tag{230}$$

Since $\cosh^{-1} u = ln(u + \sqrt{u^2 - 1})$, Eq. (227) may also be written[1]

$$C_0 = \frac{1}{2\lambda ln\left[\dfrac{D}{2r} + \sqrt{\left(\dfrac{D}{2r}\right)^2 - 1}\right]}.$$

The Capacitance of a Three-phase Line. Triangular Spacing.— Figure 42 represents a three-phase line having the three conductors unequally spaced. The plane of the paper is normal to the line and intersects the conductors in the points, A, B and C. The three unequal spacings measured in this plane are D_1, D_2 and D_3, all of which are assumed to be large compared with the radius r of the three equal conductors. Assume balanced, three-phase voltages to be impressed on the line, and let their vector values be

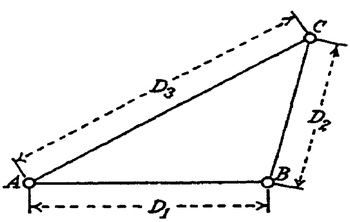


FIG. 42.—The capacitance of three-phase lines with triangular arrangement of conductors.

[1] See RUSSELL, ALEXANDER, Alternating Currents vol. 1, p 102, Eq. (8).

$$\left.\begin{aligned} E_{ab} &= E(-1 + j0) \\ E_{bc} &= E\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) \\ E_{ca} &= E\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \end{aligned}\right\} \tag{231}$$

Let the dielectric vector fluxes per unit length of line, set up on the three conductors by the impressed voltages, be represented by ψ_a, ψ_b and ψ_c. Then equating the vector impressed voltage in each loop to the drop in the corresponding dielectric circuit (the drop in the conductors themselves is assumed to be zero), the following two independent equations are obtained:

$$\left.\begin{aligned} E_{ab} &= 2\lambda\left[\psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} + \psi_c ln\frac{D_2}{D_3}\right] \\ E_{bc} &= 2\lambda\left[\psi_a ln\frac{D_3}{D_1} + \psi_b ln\frac{D_2}{r} - \psi_c ln\frac{D_2}{r}\right] \end{aligned}\right\} \text{vector volts.} \tag{232}$$

The equation for E_{ca} could, of course, have been used in place of either of the two above.

If the effect of the ground be neglected and it be assumed that no other charges are present to influence the distribution of electrical charges on the line conductors, then the neutral is at zero potential, and the vector sum of the three dielectric fluxes is zero.

Accordingly,

$$\psi_a + \psi_b + \psi_c = 0 \text{ vector lines per centimeter.} \tag{233}$$

By solving the simultaneous Eqs. (232) and (233), and substituting the known, impressed voltages in Eq. (231), the vector fluxes ψ_a, ψ_b and ψ_c may be found for any given values of the spacings D_1, D_2 and D_3. The magnitude of each of the fluxes is then readily determined from the corresponding vector expression. The potential difference between any conductor and neutral (which is also the potential of the conductor under the above assumption), may be found from Eq. (231). Dividing the flux per unit length of each conductor by the potential of the conductor will yield the corresponding capacity to neutral. For an unsymmetrical, untransposed line such as is here being considered, the capacity to neutral has a separate value for each of the three conductors, and the line is electrically unbalanced. All impor-

tant transmission lines are transposed, however, as pointed out in a succeeding paragraph. Without transposition, electrical balance in a three-phase line is possible only when the three conductors are supported at the points of an equilateral triangle. This case is next considered.

The Capacitance of a Three-phase Line. Conductors Supported at the Points of an Equilateral Triangle.—This arrangement is shown in Fig. 43. Let the three-phase vector voltages be those of Eq. (231). The expressions for the fluxes per unit length of the conductors are derived from Eq. (232) by substituting $D_1 = D_2 = D_3 = D$.

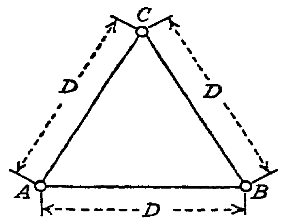


FIG 43.—The capacitance of three-phase lines with equilateral arrangement of conductors

The vector potential differences are

$$E_{ab} = 2\lambda\left(\psi_a ln\frac{D}{r} - \psi_b ln\frac{D}{r} + \psi_c ln 1\right) \text{ vector volts} \quad (234)$$

$$E_{bc} = 2\lambda\left(\psi_a ln 1 + \psi_b ln\frac{D}{r} - \psi_c ln\frac{D}{r}\right) \text{ vector volts} \quad (235)$$

and, for grounded neutral,

$$\psi_a + \psi_b + \psi_c = 0 \text{ vector lines.}$$

It is to be noted also that, for this case, due to the symmetry of arrangement, the numerical value of the flux leaving per unit length of conductor is the same for all conductors. That is, numerically, $\psi_a = \psi_b = \psi_c$

Substituting

$$\psi_a = -\psi_b - \psi_c \text{ in Eq. (234),}$$

$$E_{ab} = 2\lambda\left(2\psi_b ln\frac{D}{r} + \psi_c ln\frac{D}{r}\right). \quad (236)$$

From Eq. (235),

$$E_{bc} = 2\lambda\left(\psi_b ln\frac{D}{r} - \psi_c ln\frac{D}{r}\right). \quad (237)$$

Adding Eqs. (236) and (237),

$$E_{bc} - E_{ab} = 2\lambda\left(3\psi_b ln\frac{D}{r}\right). \quad (238)$$

Substituting the vector voltages from Eq. (231) in Eq. (238), and solving,

$$\psi_b = \frac{E(3 - j\sqrt{3})}{12\lambda ln\frac{D}{r}} \text{ vector lines} \tag{239}$$

and

$$\psi = \psi_b = \frac{\sqrt{3}E}{6\lambda ln\frac{D}{r}} \text{ lines (numerical).} \tag{240}$$

If E_0 be the numerical potential difference from one line conductor to neutral, and since $\sqrt{3}E_0 = E$, from Eq. (240),

$$\psi = \frac{E_0}{2\lambda ln\frac{D}{r}}. \tag{241}$$

The capacitance of one conductor to neutral is

$$C_0 = \frac{\psi}{E_0} = \frac{1}{2\lambda ln\frac{D}{r}} \text{ farads per centimeter} \tag{242}$$

$$= \frac{7\ 36 \times 10^{-3}}{\log_{10}\frac{D}{r}} \text{ microfarads per 1,000 ft.} \tag{243}$$

$$= \frac{3.883}{100 \log_{10}\frac{D}{r}} \text{ microfarads per mile.} \tag{244}$$

A comparison of Eqs. (244) and (213) shows that the capacitance of one conductor to neutral for a symmetrical three-phase line as illustrated in Fig. 43, is identical with that for the single-phase line having like conductors and spacing D. Comparing Eq. (241) with Eq. (207) reveals the relation of the fluxes per unit of length of conductor for the two cases. Denoting the three-phase value of Eq. (241) by ψ_3, the single-phase value of Eq. (207) by ψ_1, writing $\sqrt{3}E_0$ for E, and $\frac{D}{r}$ for $\frac{D-r}{r}$ in Eq. (213), and dividing, yields

$$\psi_3 = \frac{2\psi_1}{\sqrt{3}}. \tag{245}$$

Unsymmetrically Arranged, Transposed, Three-phase Lines. Value of C.—It has already been pointed out, in the previous

chapter, how transmission lines are transposed in order to prevent inductive interference with adjacent communication circuits. In the case of three-phase lines having unsymmetrical arrangement of conductors, transpositions are also required, in order to secure electrical balance of the three phases. For properly transposed, short, three-phase lines having unsymmetrical arrangement of conductors, it was shown that the inductance of one conductor could be expressed in terms of an equivalent spacing D', equal to the geometric mean of the individual spacings (Eq (160)). A similar relation holds for the value of C_0, the capacitance of one conductor to neutral. This relation may be shown as follows:

Since the line is assumed to be properly transposed, each of the three conductors occupies each of the three possible positions throughout a total of one-third the length of the line. Therefore, the average capacitance to neutral, per unit length of conductor, is the same for all conductors. That is,

$C_0 = C_a = C_b = C_c$ (the capacitance to neutral).

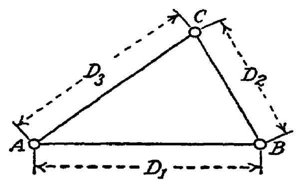


FIG. 44—Unsymmetrically arranged, transposed three-phase lines

Since each conductor occupies each of the three possible positions throughout a total of one-third the length of line, the vector voltage of phase AB, for example, may be written, by referring to Fig. 44, as

$$E_{ab} = \frac{2\lambda}{3}\left[(\psi_a - \psi_b)ln\frac{D_1}{r} - \psi_c ln\frac{D_3}{D_2} + (\psi_a - \psi_b)ln\frac{D_2}{r} - \psi_c ln\frac{D_1}{D_3} + (\psi_a - \psi_b)ln\frac{D_3}{r} - \psi_c ln\frac{D_2}{D_1}\right] \quad (246)$$

$$= \frac{2\lambda}{3}(\psi_a - \psi_b)ln\frac{D_1D_2D_3}{r^3}. \quad (247)$$

If E_a, E_b and E_c denote the vector voltages of the three conductors to neutral,

$$\left.\begin{aligned} \psi_a &= C_0E_a \\ \psi_b &= C_0E_b \\ \psi_c &= C_0E_c \end{aligned}\right\} \text{vector lines} \quad (248)$$

and

$$\left.\begin{aligned} E_{ab} &= E_a - E_b \\ E_{bc} &= E_b - E_c \\ E_{ca} &= E_c - E_a \end{aligned}\right\} \text{vector volts.} \tag{249}$$

Substituting the values of ψ_a and ψ_b from Eq. (248) and E_{ab} from Eq. (249) in Eq. (247),

$$E_a - E_b = \frac{2\lambda C_0}{3}(E_a - E_b) ln\frac{D_1D_2D_3}{r^3} \text{ vector volts,}$$

and

$$C_0 = \frac{1}{\frac{2\lambda}{3} ln\frac{D_1D_2D_3}{r^3}} \text{ farads per centimeter.} \tag{250}$$

Equivalent Spacing.—By Eq (242) the capacitance to neutral for an equilaterally spaced, three-phase line is given as

$$C_0 = \frac{1}{2\lambda ln\frac{D}{r}}.$$

Consequently, for a given conductor size, the symmetrical line which has the same capacitance to neutral as the unsymmetrical, transposed line, and which may therefore be said to be equivalent to it, must have a spacing D' of a value such that

$$2\lambda ln\frac{D'}{r} = \frac{2\lambda}{3} ln\frac{D_1D_2D_3}{r^3} \tag{251}$$

Thus,

$$\frac{D'}{r} = \sqrt[3]{\frac{D_1D_2D_3}{r^3}}$$

and

$$D' = \sqrt[3]{D_1D_2D_3}. \tag{252}$$

D' is called the equivalent spacing of the unsymmetrical line and is equal to the geometric mean of the three separate spacings. For the flat line, in which $D_1 = D_2 = \frac{D_3}{2} = D$

$$\begin{aligned} D' &= D\sqrt[3]{2} \\ &= 1\ 26D. \end{aligned} \tag{253}$$

These relations are identical with those already developed for the equivalent spacing used in computing the inductance of a transposed, unsymmetrical line, [Eq. (161)].

The Capacitance of Transmission Conductors, Including the Effect of Ground.—This problem will be solved for a three-phase

line in which the three conductors are in a horizontal plane, as in Fig. 45. The method, however, is no different for any other kind of conductor arrangement.

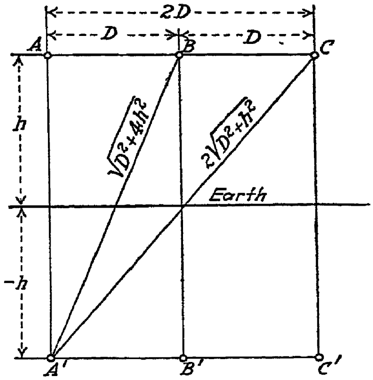


FIG. 45.—The capacitance of three-phase lines including the effect of earth

It is assumed that the earth is a zero-potential plane. Above earth, the line conductors A, B and C are supported at the height h, and, below it, are shown the equal conductors or images $A'B'C'$ at the distance $-h$ from the neutral plane. The images are assumed to carry charges, and therefore fluxes, of exactly the same amounts per unit length of conductor as the conductors themselves, but of opposite signs. This system of conductors and images will establish a plane of zero potential, coinciding with the surface of the earth, and the distribution of the dielectric flux about the conductors may be calculated by the method already employed for the conductors in which the influence of ground was neglected.

Equating the potential differences between phases to the drops in the corresponding dielectric circuits,

$$\left.\begin{aligned}
E_{ab} &= 2\lambda\Big[(\psi_a - \psi_b)ln\frac{D}{r} - \frac{1}{2}(\psi_a - \psi_b)ln\frac{D^2 + 4h^2}{4h^2} \\
&\qquad + \psi_c\Big(\frac{1}{2}ln\frac{4(D^2 + h^2)}{D^2 + 4h^2} - ln2\Big)\Big] \\
E_{bc} &= 2\lambda\Big[(\psi_b - \psi_c)ln\frac{D}{r} - \frac{1}{2}(\psi_b - \psi_c)ln\frac{D^2 + 4h^2}{4h^2} \\
&\qquad - \psi_a\Big(\frac{1}{2}ln\frac{4(D^2 + h^2)}{D^2 + 4h^2} - ln2\Big)\Big] \\
E_{ca} &= 2\lambda\Big[(\psi_c - \psi_a)ln\frac{2D}{r} - \frac{1}{2}(\psi_c - \psi_a)ln\frac{D^2 + h^2}{h^2}\Big]
\end{aligned}\right\} \quad (254)$$

Again,

$$\psi_a + \psi_b + \psi_c = 0 \quad (255)$$

$$\left.\begin{aligned}
E_{ab} &= E(-1 + j0) \\
E_{bc} &= E\Big(\frac{1}{2} - j\frac{\sqrt{3}}{2}\Big) \\
E_{ca} &= E\Big(\frac{1}{2} + j\frac{\sqrt{3}}{2}\Big)
\end{aligned}\right\} \text{vector volts.} \quad (256)$$

By substituting values of voltages from Eq. (256), dielectric flux from Eq. (255) and known values of spacings D and h in two of the three Eqs. (254), and solving, one may find the vector fluxes ψ_a, ψ_b and ψ_c. The corresponding numerical values may thus also be found. Having the latter, the capacitance to neutral follows from the relation

$$C_0 = \frac{\psi}{E_0} \text{ (numeric).} \qquad (257)$$

Double-circuit, Three-phase Lines. Value of C.—The general method of procedure to be followed in finding the capacitance of double-circuit lines is identical with that already outlined in the previous section for unsymmetrically arranged, three-phase lines.

The conductors of double-circuit lines may be arranged in a number of ways. A common arrangement is that illustrated in Fig. 31. Here each of the circuits occupies one side of the tower. The three cables of a given circuit are arranged vertically one above the other, with the middle conductor slightly offset outwardly. This arrangement of circuits will be used in the following computations illustrating the method of solving for the capacitance.

The additional assumptions made are: The two three-phase lines are operated in parallel between generating and receiving stations. The vector potentials of the conductors a and a', b and b' and c and c' are E_a, E_b and E_c with respect to the neutral, whose potential is assumed to be zero. Each of the circuits has several complete transpositions, but the corresponding conductors a and a', b and b' and c and c' are always opposite each other in the hexagonal figure formed by the six conductors on the towers.

Since the lines are transposed, each conductor occupies each of the three possible positions in rotation; the phases are properly balanced, and the capacitance to neutral is the same for all conductors.

Let the vector fluxes per centimeter of cable from a, b and c, be ψ_a, ψ_b and ψ_c, respectively. The corresponding fluxes from a', b' and c' are likewise ψ_a, ψ_b and ψ_c, since $E_a = E_{a'}$, $E_b = E_{b'}$ and $E_c = E_{c'}$.

The potential difference for phase ab of the transposed line may now be written for each of the three positions. The average value for the transposed line is

$$E_{ab} = \frac{2\lambda}{3}\left(\psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} - \psi_c ln\frac{D_2}{D_1} - \psi_a ln\frac{D_5}{D_4} - \psi_b ln\frac{D_4}{D_6} + \psi_c ln\frac{D_4}{D_3}\right.$$
$$+ \psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} + \psi_c ln\frac{D_2}{D_1} + \psi_b ln\frac{D_4}{D_6} + \psi_b ln\frac{D_5}{D_4} - \psi_c ln\frac{D_4}{D_3}$$
$$\left. + \psi_a ln\frac{D_2}{r} - \psi_b ln\frac{D_2}{r} + \psi_c ln\frac{D_1}{D_1} - \psi_a ln\frac{D_5}{D_3} + \psi_b ln\frac{D_5}{D_3} + \psi_c ln\frac{D_4}{D_4}\right)$$

$$= \frac{2\lambda}{3}\left[ln\frac{D_1^2D_4^2D_2D_3}{r^3D_5^2D_6}(\psi_a - \psi_b) - \psi_c ln\frac{D_2D_3}{D_1D_4}\cdot\frac{D_1D_4}{D_2D_3}\cdot\frac{D_1D_4}{D_1D_4}\right] \quad (258)$$

The last term in the above equation is evidently zero. Since the circuit is transposed, the capacitance to neutral is the same for all conductors, and

$$\frac{1}{C_0}(\psi_a - \psi_b) = E_a - E_b = E_{ab}$$

the vector potential difference between the conductors a and b. Hence, from Eq. (258),

$$1 = \frac{2\lambda C_0}{3} ln\frac{D_1^2D_4^2D_2D_3}{r^3D_5^2D_6}$$

and

$$C_0 = \frac{1}{\frac{2\lambda}{3} ln\frac{D_1^2D_4^2D_2D_3}{r^3D_5^2D_6}} \text{ farads per centimeter.} \quad (259)$$

Using $\log_{10}$ and expressing the capacitance in microfarads per mile, we get

$$C_0 = \frac{0\ 03883}{\log_{10}\sqrt[3]{\frac{D_1^2D_4^2D_2D_3}{r^3D_5^2D_6}}} \text{ mf. per mile.} \quad (260)$$

If we represent the equivalent spacing by D', giving D' a value such that the capacitance obtained from the equation

$$C_0 = \frac{0.03883}{\log_{10}\frac{D'}{r}}$$

is the same as that found from Eq. (260), the equivalent spacing is

$$D' = \sqrt[3]{\frac{D_1^2D_4^2D_2D_3}{D_5^2D_6}}. \quad (261)$$

The effect of the presence of two parallel circuits in close proximity is to increase the effective capacitance of each. This may be illustrated by means of an example.

Example —In the arrangement of Fig 31, let the vertical distance between adjacent conductors be 13 ft , the offset of the middle conductor from the vertical plane through a and c, be 3 ft , and let D_3, the horizontal spacing, be 22 ft. These are suitable spacings for a 132-kv. circuit. Let the conductor be 350,000-cir. mil, stranded cable having a diameter of 0 682 in Hence, its radius is 0 0284 ft The various spacings in the figure are as follows·

$$
\begin{array}{ll}
D_1 = 13\ 3 \text{ ft.} & D_4 = 28\ 2 \text{ ft.} \\
D_2 = 26\ 0 \text{ ft.} & D_5 = 34\ 0 \text{ ft.} \\
D_3 = 22\ 0 \text{ ft.} & D_6 = 28\ 0 \text{ ft.}
\end{array}
$$

By Eq. (261),

$$
\begin{aligned}
D' &= \sqrt[3]{\frac{(13\ 3)^2 \times (28\ 2)^2 \times 26 \times 22}{(34)^2 \times 28}} \\
&= \sqrt[3]{2485\ 9} = 13.55 \text{ ft.}
\end{aligned}
$$

and

$$
\begin{aligned}
C_0 &= \frac{0\ 03883}{\log_{10}\dfrac{13\ 55}{0\ 0284}} = \frac{0\ 03883}{2.6786} \\
&= 0\ 0145 \text{ mf , per mile}
\end{aligned}
$$

For a transposed single-circuit line having the same spacings as one of the lines of the figure, the equivalent spacing is

$$D' = \sqrt[3]{(13\ 3)^2 \times 26} = 16.63 \text{ ft.}$$

and

$$C_0 = 0.0140 \text{ mf per mile.}$$

Percentage increase in single-circuit capacitance, due to the proximity of the second similar circuit, is

$$\frac{500}{140} = 3.57 \text{ per cent.}$$

PROBLEMS

1. The two round conductors of a parallel-sided loop are each 0.60 in in diameter. They are separated 12 ft. between centers A potential difference of 100 kv. is impressed across the loop. In a sectional view normal to the plane of the loop, draw the equipotential lines for each 5 kv. of potential difference.

2. For the circuit of Problem 1, in the sectional view, draw 5 dielectric lines of force, so chosen that each 2 adjacent lines shall include between them one-eighth of the flux per centimeter of conductor. Let the straight line joining the two conductors be one of the lines as well as the axis of symmetry of the completed figure.

3. Two equations are given for the capacity of two round parallel wires, namely,

$$(a)\ C_0 = \frac{3.883}{100 \log_{10} \dfrac{D - r}{r}} \text{ mf. per mile}$$

$$(b)\ C_0 = \frac{8\ 948}{100 \cosh^{-1} \dfrac{D}{2r}} \text{ mf per mile.}$$

Calculate the capacity of two parallel conductors each 1 in. in diameter, by each of the above equations, for various separations between centers from slightly more than 1 in. up to 36 in. Between these limits calculate the percentage of error made when using the approximate equation and plot error percentage *vs* separation in inches.

4. A three-phase, 60-cycle, 110-kv line is built of 300,000-cir mil cables whose diameters are 0 631 in The conductors are supported at the three points of a triangle whose sides are 12, 12, and 20 ft The line is transposed to equalize the phases. What is the charging current per phase, per mile of the line?

5. A double-circuit, 150-kv., 60-cycle, three-phase line has six conductors, each of diameter 0.773 in., supported vertically on each side of the tower line in a manner similar to that shown in Fig 31. The vertical separation between adjacent conductors is 11 ft 6 in, the middle conductor is offset from the vertical plane, and away from the tower, by 4 ft 6 in, and the separation between the planes of the two circuits is 14 ft 6 in The mean distance to earth of lowest conductor is 30 ft Beginning at the upper left-hand corner of the hexagonal figure representing the conductor arrangement, and reading around the figure counter-clockwise, the conductors lie in the order a, b, c, c', b', a'. The conductors a and a', b and b' and c and c' are connected to the busses A, B and C, respectively.

If the lines are transposed to balance phases, calculate (*a*) the capacity to neutral of each line, neglecting the effect of ground, (*b*) the capacity to neutral of each line including the effect of ground. Assume the neutral plane at the earth's surface.

CHAPTER V

CORONA

Description.—Our interest here in the phenomenon of corona concerns its appearance on the parallel conductors of a transmission-line circuit, suspended from insulators in air, a medium of unit permittivity. Under conditions of low-potential difference impressed on the conductors, air is practically a perfect insulator. If the impressed potential difference is alternating, the dielectric flux is alternating, giving rise to a *displacement current* in the dielectric circuit and the usual *charging current* in the conductors. The energy flow in the circuit is purely reactive, since no energy is lost in the dielectric.

As the voltage is continually raised, the potential gradient about the conductors is correspondingly increased until, at the critical gradient of about 30 kv. per centimeter, the air in the immediate vicinity of the conductor becomes conducting. If viewed in the dark, a violet glow becomes visible, a hissing sound is heard, the conductor tends to vibrate, and, if conditions are favorable, the presence of ozone may be detected by its characteristic odor. The phenomenon, the evidences of which have just been recited, is called *corona*. The minimum voltage at which the glow is observed is called the *critical visual corona voltage.* If the conductors are rough or dirty, the potential gradient will not be uniform for all parts of the conductor, but will be greatest at the roughened surfaces where the curvature is greatest. Corona will appear first at these points, and will continue there in intensified form, giving the conductor the appearance of lumpiness. A wattmeter placed in the electrical circuit will indicate a loss of power, showing that energy is being dissipated in the leakage circuit. The lost energy appears in various forms such as heat, light, chemical energy and mechanical energy of sound and vibration generally. A measure of the charging current under corona conditions indicates an apparent increase in the capacitance of the line. The explanation usually given has been that the corona increases the effective diameter of the

conductor to the outer boundary of the corona envelope. An increase in the capacitance of the conductor would naturally follow. Later experiments[1] make it appear that no actual increase in capacitance occurs, but that the apparent increase is due to the presence of harmonic currents introduced by the corona cycle.

If the voltage be raised to still higher values, the luminous glow will reach out farther and farther from the conductors until, finally, the insulation of the air is broken down completely and an arc passes between the conductors. This voltage, which for widely spaced conductors is considerably higher than the visual corona voltage, is called the *sparkover voltage.* If the voltage is held at this value, the sparks or arcs may be intermittent, corona reappearing between successive discharges. For small, closely spaced conductors, sparkover may appear without previous formation of corona.

Theory of Corona Formation.—The formation of corona may be accounted for on the basis of the electron theory somewhat as follows: Under ordinary conditions, even when free from electrical stress, air contains a certain number of free electrons; that is, it is slightly ionized When a potential difference is gradually applied between conductors, a potential gradient is established in the space about the conductors and between them. In response to this force the electrons acquire a uniformly accelerated motion, and in very short distances attain high velocities. The velocity acquired in a given time depends upon the mass of the electron, the size of its charge, and the potential gradient of the field causing the acceleration.

The moving electrons collide with one another, and with the larger and more slowly moving, neutral molecules. The *average distance* which the electrons travel before making a collision, called the *mean free path,* depends upon the number of molecules per unit volume of the air, and thus upon the temperature and barometric pressure. When the potential gradient reaches a value of about 30 kv. per centimeter, assuming standard conditions of temperature and pressure, the electrons acquire sufficient velocity in moving a distance equal to their mean free path, to dislodge one or more electrons from a neutral molecule when

[1] Gardner, Murray F., "Corona Investigation on an Artificial Line," *Proc.*, A. I. E. E., p. 183, August, 1925.

Peek, F. W., Jr., "Voltage and Current Harmonies Caused by Corona," *Trans* , A. I. E. E., p 1155, 1921

colliding with it. This results in additional free electrons, and molecules deficient in electrons, called ions. Both of these in turn are accelerated, resulting in increasing numbers of new collisions, free electrons and ions. The numbers of these present in a given volume thus rapidly accumulate, until saturation is reached, the insulating properties of the air are destroyed, the air becomes conducting, and corona forms or a spark passes between the conductors.

It has been found experimentally that the constant gradient of approximately 30 kv. per centimeter is the gradient at which the cumulative effect of ionization will cause corona to form. This, however, is not the gradient at the surface of the conductor, but at a distance from the center of a conductor whose radius[1] is r of $x = (r + 0.301\sqrt{r})$. The gradient at the surface of the conductor, designated as G_v, the visual corona gradient, is somewhat higher, and is given by the equation

$$G_v = G_0\left(1 + \frac{0.301}{\sqrt{r}}\right)$$

The observed results are explained by the fact that energy must be transferred from the electric circuit to the surrounding air before corona can form. The amount of energy required is the sum of all the energies possessed by the moving ions and electrons in the vicinity of the conductor when saturation is reached, and the air becomes conducting. If the gradient of the surface of the conductor were but 30 kv. per centimeter, the gradients at greater distances from the center would be less, and therefore not sufficient to cause the cumulative ionization required.

Experimental Investigation.[2] Factors Influencing Corona.—As a result of extensive experiments, and particularly as a result of the very complete investigation carried out under the supervision of F. W. Peek, Jr., at the General Electric Company

[1] PEEK, F. W., JR., "Law of Corona," *Proc.*, A. I. E. E., June, 1911.

[2] The present body of knowledge pertaining to the subject of corona is the result of many experiments performed by various investigators and reported in the electrical journals. These experiments began many years ago and are still being reported from year to year. Among those who have made important contributions to this knowledge are: Mershon, Ryan, Steinmetz, Peek, Whitehead and Harding. For results of their experiments, see *Trans.*, A. I. E. E., 1908 to 1924; also, PEEK, F. W., JR., "Dielectric Phenomena in High Voltage Engineering."

about 1910, the factors which influence the formation of corona and the laws which govern it, are now fairly well established Among the conclusions reached, the following, applying to the parallel conductors of a transmission line, are important to the engineers:

1. *Conductor, Diameter and Spacing.*—The stress at any point in a dielectric is measured by its corresponding potential gradient, G.

From Eq. (205), by substituting r for x, we get

$$_{\max}G = 2\lambda\psi\left(\frac{1}{r} + \frac{1}{D - r}\right)$$

and by slightly rearranging Eq. (207) one may write

$$\psi = \frac{E_0}{2\lambda ln\frac{D}{r}}$$

Eliminating λ from the equation for maximum gradient and simplifying yields the potential gradient at the surface of one of two parallel wires spaced far apart, as in transmission-line practice. It is, approximately,

$$_{max}G = \frac{de}{dx} = \frac{E_n}{rln\frac{D}{r}}$$

where r = radius of the conductor in centimeters.
D = spacing between conductors in centimeters.
E_n = potential difference from conductor to neutral.

Since the gradient varies inversely as the distance from the center of the conductor, the above is the maximum gradient. If the voltage to neutral E_n, be raised to the critical value E_v, at which visual corona appears, the corresponding gradient G_v is then

$$G_v = \frac{E_v}{rln\frac{D}{r}} \qquad (262)$$

$$= \frac{0.4343E_v}{r\log_{10}\frac{D}{r}}.$$

It is apparent from the above that both the conductor diameter and the spacing are factors influencing the surface gradient, and therefore the voltage at which visual corona starts.

Experiment shows that at sea level under standard conditions of temperature and barometric pressure, namely 25° C. and 76 cm. of mercury barometer, respectively, visual corona appears on parallel, round conductors at varying surface gradients. At a distance from the center of the conductor of

$$x = r + 0.301\sqrt{r}\text{ cm.} \tag{263}$$

the gradient is, however, constant and equal to approximately 30 kv. per centimeter. For sine waves of e.m f. this is the maximum value. The corresponding gradient in terms of effective volts is 21.1 kv. per centimeter or 53.6 kv. per inch. This constant gradient G_0, is less than the surface gradient G_v. The two gradients are related as shown in Eq. (264). That is,

$$G_v = G_0\left(1 + \frac{0.301}{\sqrt{r}}\right). \tag{264}$$

It is apparent from Eq. (264) that, to start visual corona, small conductors require a higher surface gradient than large ones.

By substituting the value of G_v from Eq. (262) in Eq. (264), writing for G_0 its equivalent, and solving for E_v, there results:

$$E_v = 21.1\left(1 + \frac{0\ 301}{\sqrt{r}}\right)rln\frac{D}{r}\text{ kv. effective to neutral.} \tag{265}$$

This is the equation for the voltage to neutral required to start visual corona in fair weather under standard conditions of temperature and pressure.

2. *Density of Air.*—As already intimated, the air density is a factor affecting the voltage at which corona begins. This factor, denoted δ, is unity at 760-mm pressure and 25° C. temperature. For other temperatures or barometric pressures, it is the fraction

$$\delta = \frac{3.92b}{273 + t} \tag{266}$$

where b = barometer reading in centimeters of mercury
t = temperature in degrees Centigrade.

The factor is directly proportional to the barometric pressure and inversely proportional to the absolute temperature.

Obviously also, it decreases with increase in altitude, other conditions remaining unchanged.

3. *Roughness of Conductor Surface.*—On conductors with rough surfaces, corona starts at voltages lower than for conductors with smooth surfaces. The reason for this is apparent. For a rough spot has the effect of increasing the curvature of the surface at that point, and thus of increasing the local, potential gradient. A factor m, called the *roughness factor*, is therefore introduced to give proper weight to the influence of the condition of conductor surface upon the corona voltage. Thus, the corona starting voltage becomes proportional also to m.

Deposits on the conductor surface, such as moisture, snow or sleet, produce a roughness effect and lower the corona voltage.

Corona Voltage and Altitude Roughness Factors Included.—Equation (265) gives the visual corona voltage to neutral for standard conditions of temperature and pressure and for a smooth conductor. For other conditions the correction factors δ and m are introduced, and Eq. (265) becomes

$$E_v = 21.1m_v\delta r\left(1 + \frac{0.301}{\sqrt{\delta r}}\right)ln\frac{D}{r} \text{ kv. effective to neutral.} \quad (267)$$

The corresponding disruptive critical voltage is

$$E_0 = 21.1m_0\delta r ln\frac{D}{r} \text{ kv. effective to neutral.} \quad (268)$$

In Eqs. (267) and (268) c g s. units and natural logarithms are used When logarithms to the base 10, degrees Fahrenheit and inches are used, these equations become

$$E_v = 123m_v\delta r\left(1 + \frac{0.189}{\sqrt{\delta r}}\right)\log_{10}\frac{D}{r} \text{ kv effective to neutral} \quad (269)$$

$$E_0 = 123m_0\delta r \log_{10}\frac{D}{r} \text{ kv. effective to neutral} \quad (270)$$

where

δ = air density factor

$= \dfrac{17\ 9b}{459 + t}$

= 1 at 77° F. and 29.9 in., barometer

t = temperature in degrees Fahrenheit

b = barometer reading in inches of mercury

r = radius of conductor in inches

D = interaxial spacing in inches

m_v = irregularity factor for E_v
m_0 = irregularity factor for E_0

$$G_0 = \frac{123}{2.302} = 53.6 \text{ kv. per inch.}$$

The irregularity factor has the approximate values given below:

m_0 = 1 for polished wires
= 0.98 to 0.93 for roughened or weathered wires
= 0.87 to 0.83 for seven-strand cable
= 0.85 to 0.80 for 19-, 37- and 61-strand, concentric, lay cables

A value of $m_0 = 0.68$ for a piece of new, 49-strand, 500,000-cir. mil rope lay cable, was reported by Wilkins.[1] Three years later a test of the line showed $m_0 = 0.72$. While these low values are no doubt due to the use of rope lay cable, they clearly illustrate the improvement to be effected in the value of m_0 with use. This is due to the disappearance of the roughened surfaces and irregularities by oxidation. They also argue powerfully against the use of rope lay cables for high-voltage lines. For large, concentric lay cables from which the rough points left by the manufacturing and handling processes have been weathered away by some years of use, a value of $m_0 = 0.85$ is probably a safe value to use. Since the accuracy of the results to be expected from the loss equation, are largely dependent upon the accuracy with which m_0 is known, the importance of properly determining this factor is apparent.

Calculation of Maximum Potential Gradients.—The maximum potential gradient is at the surface of the conductor, and is proportional to the dielectric flux density. The latter, in turn, is equal to the dielectric flux per unit length of conductor divided by the circumference of the conductor; or

$$G = \frac{2\lambda\psi}{r}$$

where

$$\lambda = \frac{v^2}{10^9 k}.$$

Thus, to compare the maximum gradients produced at the surfaces of conductors of a given diameter, but having various spacing arrangements, it is only necessary to compare their dielectric

[1] Wilkins, Roy, "Corona Loss Tests on 202-mile, 60-cycle, 220-kv., Pitt-Vacca Transmission Line," *Proc.*, A. I. E. E., December, 1924.

fluxes. It will be assumed in all cases that the spacing is large and that the error made in substituting D for $D - r$ is negligible.

1. *Single-phase Line.*—From Eq. (207), the dielectric flux, per unit length of conductor for a single-phase line, is readily seen to be

$$\psi = \frac{0.4343E_n}{2\lambda \log_{10}\frac{D}{r}} \tag{271}$$

and the corresponding maximum gradient is

$$_{\max}G = \frac{0.4343E_n}{r \log_{10}\frac{D}{r}} \tag{272}$$

2. *Three-phase Line, General.*—Consider a three-phase line having loops of widths D_1, D_2 and D_3, whose corresponding vector voltages are E_{ab}, E_{bc} and E_{ca}, producing, respectively, per unit length of line, the dielectric vector fluxes ψ_a, ψ_b and ψ_c. The voltage of a loop is

$$E = \int_r^{D-r} \psi \cdot dS.$$

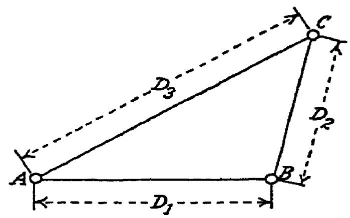


FIG. 46.—Potential gradients on three-phase lines

Writing these integrals for two of the loops, and replacing $D - r$ by D,

$$E_{ab} = 2\lambda\left[\psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} - \psi_c ln\frac{D_3}{D_2}\right] \text{ vector volts} \tag{273}$$

$$E_{bc} = 2\lambda\left[\psi_a ln\frac{D_3}{D_1} + \psi_b ln\frac{D_2}{r} - \psi_c ln\frac{D_2}{r}\right] \text{ vector volts} \tag{274}$$

$$E_{ca} = 2\lambda\left[-\psi_a ln\frac{D_3}{r} + \psi_b ln\frac{D_1}{D_2} + \psi_c ln\frac{D_3}{r}\right] \text{ vector volts.} \tag{275}$$

Also,

$$0 = \psi_a + \psi_b + \psi_c \tag{276}$$

since the vector sum of all the dielectric fluxes is zero. The vector voltages may be written,

$$E_{ab} = E(1 + j0)$$

$$E_{bc} = E\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$E_{ca} = E\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right).$$

If the spacings D_1, D_2 and D_3 and the radius of the conductor are known, the values of ψ_a, ψ_b and ψ_c for the corresponding three-phase line may be found.

3. *Three-phase Line, Equilateral Spacing.*—For this case $D_1 = D_2 = D_3 = D$. Substituting these values in Eqs. (273) and (274), and solving for ψ_a, ψ_b and ψ_c from Eqs. (273), (274), and (276)

$$\psi_b = \frac{E(-3 + j\sqrt{3})}{2\lambda \cdot 6 ln\frac{D}{r}} \text{ vector lines} \tag{277}$$

where E is the numerical value of the line voltage

and

$$\psi_b = \frac{0.4343 E_n}{2\lambda \log_{10} \frac{D}{r}} \text{ lines absolute,} \tag{278}$$

whence

$$G_b = \frac{0.4343 E_n}{r \log_{10} \frac{D}{r}} \text{ kv. per inch.} \tag{279}$$

Equations (278) and (279) are the same as the corresponding equations for the single-phase line. By symmetry, the absolute values of ψ_a and ψ_c are each equal to ψ_b. Comparing Eqs. (278) and (279) with Eqs. (271) and (272) shows that, for conductors of like diameters and spacings, the maximum gradient for the three-phase case bears the ratio of $2 \div \sqrt{3}$ to the maximum, single-phase gradient for equal line voltages. They are equal for equal voltages to neutral.

4. *Three-phase Line, Flat Spacing.*—Let the conductors lie in a single plane, and let $D_1 = D_2 = \frac{D_3}{2} = D$. Substituting these values in Eqs. (273) and (274), and solving as before, the flux per unit length of middle conductor is

$$\psi_b = -\frac{E(3 - j\sqrt{3})}{2\lambda \cdot 2\left(3 ln\frac{D}{r} - ln 2\right)} \text{ vector lines} \tag{280}$$

and

$$\psi_b = \frac{E}{2\lambda\left(\sqrt{3} ln\frac{D}{r} - \frac{1}{\sqrt{3}} ln 2\right)} \text{ lines absolute} \tag{281}$$

whence

$$G_b = \frac{E}{r\left(\sqrt{3}ln\frac{D}{r} - \frac{1}{\sqrt{3}}ln2\right)} \text{ kv. per inch} \qquad (282)$$

$$= \frac{0\ 4343E_n}{r\left(\log_{10}\frac{D}{r} - 0.333\log_{10}2\right)} \text{ kv. per inch.} \qquad (283)$$

In a similar manner, ψ_a may be calculated, after which ψ_c may conveniently be found by substituting the value of ψ_b from Eq. (281) in Eq. (276), after substituting for ψ_a in the latter. The scalar values of ψ_a and ψ_c are equal, from symmetry. The value of ψ_c is

$$\psi_c = +\frac{E}{2\lambda}\left[\frac{3 - j\sqrt{3}}{12ln\frac{D}{r} - 4ln2} - \frac{1 + j\sqrt{3}}{4\left(ln\frac{D}{r} + ln2\right)}\right] \text{ vector lines.} \qquad (284)$$

The absolute values of ψ_a and ψ_c are obtained by substitution in Eq. (284), simplifying to get the resultant vector and subsequently evaluating. Table 7 gives the maximum gradients for the middle and outside conductors in percentage of the maximum gradient for the equilateral three-phase line and for given values of D and r.

TABLE 7.—PERCENTAGE RATIOS $\frac{G_F}{G_\Delta}$ FOR THREE-PHASE LINES AND FOR FLAT SPACING ($D_1 = D_2 = \frac{D_3}{2} = D$)

	Per cent ratio $G_F \div G_\Delta$ for three-phase line, flat spacing	
$\log_{10}\frac{D}{r}$	Outside conductors	Middle conductors
1.00	86 7	111 2
1.25	88 4	108 7
1 50	89 8	107.2
1 75	91 0	106 1
2 00	91 9	105 3
2.25	92 6	104.7
2.50	93 2	104 2
2.75	93 5	103.8
3 00	94 0	103.4
3.25	94 0	103 2

G_F = maximum gradient for flat spacing
G_Δ = maximum gradient for equilateral triangle type of spacing.

Corona Loss.[1]—Energy loss due to corona begins at the *disruptive critical voltage* of Eq. (270), a value somewhat below the visual voltage of Eq. (269). Peek found, as a result of a long series of tests on an experimental transmission line, that the loss is proportional to the square of the voltage in excess of the disruptive, critical value, and is influenced by several other factors. For single-phase lines and three-phase lines having equilateral spacings, the fair-weather loss is

$$P = \frac{390}{\delta}(f + 25)\sqrt{\frac{r}{D}}(E_n - E_0)^2 \times 10^{-5} \text{ kw. per mile of one conductor.} \quad (285)$$

where f is the frequency employed in cycles per second and E_n and E_o are the impressed and critical, disruptive kilovolts to neutral, respectively. All other symbols in this equation are in the inch system of units as previously defined.

To approximate the loss under storm conditions, E_0 is taken as 0.80 of its corresponding fair-weather value.

When the spacings employed on three-phase lines are not equal, as for example when flat, horizontal spacings or triangular spacings with unequal sides are used, Eq. (285) gives results which are in error. For the flat spacing, corona starts on the middle conductor at a voltage approximately 4 per cent below the critical value for the equilateral arrangement having the same spacing D, while on the outer conductors it starts at a voltage about 6 per cent higher (Table 7). The reason for the differences is apparent, since the gradient at the middle conductor must necessarily be higher than that for the outside ones.

The gradient is the determining factor, and, since in unequally spaced lines it will not be the same for all conductors, it is desirable to express the loss equation in terms of gradients rather than voltages, to take care of loss calculations for such cases. This is done by substituting in Eq. (285) the corresponding expression for voltage, or

$$E_n = 2.302Gr \log_{10} \frac{D}{r} \text{ from Eqs. (204) and (207)} \quad (286)$$

and

$$E_0 = 2.302G_0m_0\delta r \log_{10} \frac{D}{r} \text{ from Eq. (270).} \quad (287)$$

[1] PEEK, F. W., JR., "Law of the Corona," *Trans.*, A. I. E. E , "Dielectric Phenomena in High-voltage Engineering," p. 204 and following, 1920.

Equation (285) then becomes

$$P = \frac{2070}{\delta}(f + 25)\sqrt{\frac{r}{D}}\left(\log_{10}\frac{D}{r}\right)^2 r^2(G - m_0\delta G_0)^2 \times 10^{-5}$$

kw. per mile of one conductor. (288)

This equation, like Eq. (285), employs the inch units of length Equation (288) may be used to calculate the loss on each conductor separately as soon as the gradients are known. For flat spacings the factors of Table 7 may be used to find the gradients.

Corona Loss Calculation.—Let it be required to estimate the corona loss per mile of a transmission line for which the following data apply:

Cable diameter = 0.681 in. (concentric lay cable)
Number of strands = 37
Spacing, flat $D_1 = D_2 = \frac{D_3}{2} = D = 16$ ft.

Frequency = 60 cycles per second
Roughness factor $m_0 = 0.83$
Altitude = 1,000 ft
Fair-weather value of $\delta = 0.95$
Disruptive, critical gradient $G_0 = 53.6$ kv. per inch
Applied voltage = 97 kv. to neutral

$$E = 2.302 G_0 m_0 \delta r \log_{10}\frac{D}{r}$$

$$= 0.302 \times 53.6 \times 0.83 \times 0.95 \times 0.340 \times 2\,752$$

= 91,000 volts to neutral for equilateral spacing of 16 ft.

$$G = \frac{0\,4343 E_n}{r \log_{10}\frac{D}{r}}, \text{ from Eq (262)}$$

$$= \frac{0.97 \times 0.4343}{0\,340 \times 2.752} = 45 \text{ kv. per inch for three-phase, equilateral spacing.}$$

From Table 7,

$G_a = G_c = 0.935 \times 45 = 42+$ kv. per inch for outside conductors.
$G_b = 1.038 \times 45 = 46.6$ kv. per inch for middle conductor.

By Eq. (288),

$$P = \frac{2{,}070}{\delta}(f + 25)\sqrt{\frac{r}{D}}\left(\log_{10}\frac{D}{r}\right)^2 r^2(G - m_0\delta G_0)^2 \times 10^{-5} \text{ kw. per mile of one conductor.}$$

$$= \frac{2{,}070 \times 85}{0.95}\sqrt{\frac{0\,340}{192}}(2.75)^2(0.340)^2(G - 0.83 \times 0.95 \times 53.6)^2 \times 10^{-5}$$

$$= 0.06829(G - 42\,3)^2$$

From Table 7, for $\log_{10} \frac{D}{r} = 2.75$, as before,

$G_a = 42$ kv. per inch for two outside conductors.
$G_b = 46.6$ kv. per inch for mid-conductor.

There is no corona on the outside conductors since $G_a = G_c < (42.3 = m_0G_0\delta)$.
The loss per mile on the mid-conductor is

$$\begin{aligned} P &= 0.06829(48.6 - 42.3)^2 \\ &= 1.26 \text{ kw. per mile.} \end{aligned}$$

Corona as a Factor in Line Design.—A study of the economics of transmission-line design shows that, generally speaking, the most economical line for a given project is the one having minimum, annual, fixed charges. Such a line will fulfill the requirements of Kelvin's Law and will operate at or near corona voltage. What the exact value of the voltage should be, that is, whether it should be a little above or a little below the disruptive, critical value for average, fair-weather conditions, is a question which is difficult to determine. Consideration must be given to the fact that seasonal temperature changes may be sufficient to cause a variation of critical, disruptive voltage of perhaps 20 per cent between winter and summer. Variations in altitude for different sections of the line and the prevalence or absence of storms are other factors which should be given proper weight. From the purely economic standpoint alone, it would seem that the correct operating voltage should be slightly under the critical value for fairweather and summer temperatures. A choice of this voltage would eliminate corona under all fair-weather, normal, operating conditions and still would leave incidental advantages to be gained under normal voltages.

Lines which are built to operate near the critical voltage not only are the most economical, but they have additional protection against the destructive effects of high-voltage, high-frequency surges occasioned by lightning, switching and other transient disturbances. It has been pointed out by Peek, Whitehead and others, that surges, due to suddenly applied over-voltage resulting in corona formation, are greatly attenuated in traveling over a comparatively short stretch of line. The energy of the surge is dissipated in the corona leakance, and the excess voltage is thereby reduced to a safe value. By normally operating

close to the critical voltage, the maximum advantage is gained from this protective feature.

PROBLEMS

1. A 60-cycle, three-phase line is built of 250,000-cir. mil, concentric lay cable, for which the roughness factor is 0 85. The conductors are spaced 12 ft apart at the vertices of an equilateral triangle. Assuming δ to vary with altitude, as in Table 24, at what voltage will corona begin to form on this line at 2,000 ft. altitude? At sea level?

2. For the line in problem 1, at what voltage would the corona loss be 5 kw. per mile at 2,000 ft. altitude?

3. Assuming an equilateral arrangement of conductors and a line free from corona loss, what is the maximum allowable voltage for a 0000 solid conductor line, whose spacing of conductors is 11 ft. between centers? $m_0 =$ 0 93. $\delta = 0\ 96$.

4. The three equal cables of a 165-kv, three-phase line lie in a horizontal plane, and are separated by 20 ft. between centers. The irregularity factor is 0 83 and the value of δ is 0.94. What is the minimum conductor diameter that may be used on this line if corona loss is to be avoided?

5. If we assume (*a*) that for single-circuit lines with flat spacing, the separation of conductors in inches, demanded by good practice, is given by the equation.

$$D' = 0.002E_n \text{ (Eq. 528)}$$

and (*b*) that near sea level the voltage E_n to neutral in k v is 240 times the conductor radius in inches, what is then the minimum permissible separation of the conductors of a 165-kv. line having no corona? The conductors are arranged in a plane and the elevation of the line is sea level.

CHAPTER VI

INDUCTIVE INTERFERENCE

When power circuits and communication lines are in close proximity, and particularly when they parallel one another for considerable distances, potential differences are induced between the wires of the communication circuits, due to the varying magnetic and dielectric fields set up by the power circuits. These potential differences establish currents in the wires, instruments, relays and other devices that compose the essential elements of the communication system. Since the amount of current required to produce an audible sound in a telephone receiver is very small indeed, even feeble currents may be troublesome, especially if their frequencies lie within the range of voice frequencies. In telephone systems the induced currents cause a humming noise in the receiver, thereby annoying the user and reducing the intelligibility of conversation, while in telegraph circuits the disturbance may cause chattering of relays, reduced speed of transmission and impaired clearness of signaling. Disturbances in telegraph circuits are due largely to the fundamental frequency, while the harmonics are the source of the disturbances most affecting telephonic communication. The situation thus created by the proximity of power and communication circuits constitutes the problem of *inductive interference*. Since power circuits cannot be divested of their dielectric and magnetic fields and communication circuits cannot be operated with complete satisfaction when indiscriminately immersed in them, a satisfactory solution of the problem can only be had as a result of the closest cooperation between the power and communication interests. The problem has received a good deal of attention at the hands of engineers and considerable literature on the subject is available. Perhaps the most extensive single source is to be found in the report of the California Railroad Commission entitled "Inductive Interference."

Electromagnetically-induced Voltages. 1. *Single-phase Line.* In Chap. III is discussed in some detail how and to what

extent alternating currents in one conductor produce varying magnetic fields that reach out and link other adjacent conductors, thereby inducing potential differences in them. These potential differences, even though very small as compared with those impressed upon power circuits, may nevertheless produce a disturbance in a nearby telephone circuit owing to the sensitivity of the instruments used.

To illustrate how electromagnetically-induced potential differences are set up, assume a single-phase power circuit ab, and a nearby, parallel telephone circuit mn, m being the near conductor and n the more remote one. Let the power circuit carry a sine-wave current whose value is I absolute units. If the current in a be taken as positive, that in b is negative, and their sum is zero; that is,

$$I_a = I_b = 0.$$

Representing the distances from a to m, a to n, etc , by the appropriate letters am, etc , the flux linking the telephone circuit per centimeter of loop is

$$\begin{aligned} \phi_{mn} &= 2I\left(ln\frac{an}{am} - ln\frac{bn}{bm}\right) \\ &= 2Iln\frac{an \cdot bm}{am\ bn} \text{ linkages.} \end{aligned} \tag{289}$$

Neglecting the effect of the telephonic currents on the power circuit (they are extremely small), the coefficient of mutual induction per centimeter of loop is

$$M = \frac{\phi_{mn}}{I} = 2ln\frac{an \cdot bm}{am \cdot bn}$$

and the induced voltage per centimeter of loop is

$$\begin{aligned} E_{mn} &= j\omega MI = j\omega\phi_{mn} \\ &= 2j\omega Iln\frac{an \cdot bm}{am \cdot bn} \text{ abvolts per centimeter.} \end{aligned} \tag{290}$$

Only that part of the flux from each conductor which threads the loop of the telephone circuit induces any resultant potential difference, for all flux lines which link both conductors of a circuit induce therein equal voltages that are oppositely directed around the loop. The amount of flux through the loop depends upon the current and upon the difference of the logarithms of the

distances between the disturbing wires and the telephone loop, as in Eq. (289). The induced voltage is proportional to the flux (and hence to the current), and to the frequency of the inducing current. As the distance between the power wires is reduced, the fraction whose logarithm is involved in Eq. (290) approaches zero, as does also the induced potential difference.

2. *Three-phase Line.*—If the disturbing circuit is a three-phase power line, the magnetic fields due to the currents in the three phases all induce potential differences in the loop of the communication circuit. The resultant potential difference induced is the vector sum of the individual potential differences. Thus if I_a, I_b and I_c are the known vector sine-wave currents in the three line conductors a, b and c of the power circuit, and m and n are the nearer and the farther wires of the communication circuit respectively, the induced vector potential difference in the communication circuit loop is

$$\begin{aligned} E_{mn} &= j\omega\phi_{mn} \\ &= j\omega\left[2I_a ln\frac{an}{am} + 2I_b ln\frac{bn}{bm} + 2I_c ln\frac{cn}{cm}\right] \text{ vector abvolts per centimeter.} \qquad (291) \end{aligned}$$

When harmonics are present, the induced potential difference due to each is separately considered. If, per unit of induced potential difference in the given circuit, all harmonics may be considered as being approximately equal in their disturbing effects, the effect due to all harmonics may be taken as approximately proportional to the effective value of the total induced potential difference, that is, to the square root of the sum of the squares of the potential differences due to the individual harmonics.

If the three-phase currents are balanced, their vector sum is zero. The three vector currents may then be represented as

$$\begin{aligned} I_a &= I_0(1 + j0) \\ I_b &= -\frac{I_0}{2}(1 + j\sqrt{3}) \\ I_c &= \frac{I_0}{2}(-1 + j\sqrt{3}). \end{aligned}$$

Substituting these currents in Eq. (291), the induced potential difference in mn is

$$E_{mn} = j\omega\left(2I_0 ln\frac{an}{am} - I_0 ln\frac{bn}{bm} - I_0 ln\frac{cn}{cm} - jI_0\sqrt{3}ln\frac{bn}{bm} + jI_0\sqrt{3}ln\frac{cn}{cm}\right)$$

$$= I_0\omega\left(jln\frac{\overline{an}^2 \cdot bm \quad cm}{\overline{am}^2 \cdot bn \cdot cn} - \sqrt{3}ln\frac{cn \quad bm}{cm \cdot bn}\right) \text{ vector abvolts per centimeter of loop} \quad (292)$$

or, in amount, the potential difference is

$$E_{mn} = 370.56 \times 10^{-6} I_0\omega \times \sqrt{\left(\log_{10}\frac{\overline{an}^2 \cdot bm \cdot cm}{\overline{am}^2 \cdot bn \quad cn}\right)^2 + 3\left(\log_{10}\frac{cn \cdot bm}{cm \cdot bn}\right)^2} \text{ abvolts per mile of loop.} \quad (293)$$

Residual Currents in Unbalanced Three-phase Lines.—The currents in a three-phase line are said to be balanced when their vector sum is zero. Conversely, in an unbalanced line the vector sum of the three currents yields a resultant current which is not zero This resultant current is called the *residual current.* A convenient method[1] of calculating the inductive effects of such currents is to deal with their equivalent, balanced, three-phase and single-phase components as discussed below.

The three line currents of an unbalanced three-phase line may be resolved into (*a*) three equal line currents differing in phase by 120° (balanced currents); (*b*) three equal, residual currents, each of which is one-third of the residual current, as defined above, flowing in a loop of which one side is the neutral conductor (or ground return) and the other consists of the three line wires in multiple; and (*c*) a single-phase current flowing out in one line conductor and returning in another. A vector diagram showing these relations is given in Fig. 47. The inductive effects of unbalanced currents may be calculated by dealing with these components in place of the unbalanced currents themselves, using the methods already described.

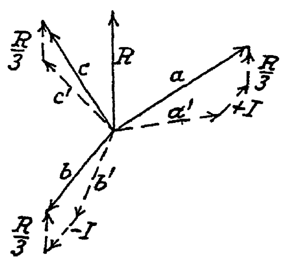


FIG. 47.—Residual currents in unbalanced three-phase lines

[1] "Inductive Interference between Elctric Power and Communication Circuits," California Railroad Commission, p. 120.

In the diagram a, b and c are the unbalanced three-phase currents, equivalent to the three balanced currents a', b' and c', plus the single-phase residual R, plus the single-phase current I circulating in the leads A and B.

The effect of the residual currents is, in general, of far greater importance than that of the balanced, three-phase currents, because the ratio of the distances from the disturbing to the disturbed conductors, shown in Eq. (293), is usually greater for the residuals.

Electrostatically-induced Voltages.—Electrostatically-induced potentials are set up in conductors occupying any space in which there is a dielectric field. In the space about a power circuit, each power conductor sets up a varying dielectric field which is at all times proportional to the relative potential of the conductor. In such a field the two conductors of a communication line paralleling a power line will, in general, be subjected to fields of unequal intensities; the electrical potentials of the conductors will be unequal, and thus there will be a difference of potential between them. There will also be a difference of potential between the wires and ground, which, in general, is many times as large as that between the two wires of a circuit.

The electrostatically-induced potential differences may be calculated by the methods of Chap. IV. Let a, b and c be the three conductors of an untransposed, three-phase power circuit and let m and n be the two conductors of a communication circuit paralleling the power line, m being the near conductor as before. Let r be the radius of each of the three equal power cables and represent the distances from each power cable to the wires of the communication circuit by the appropriate letters as am, bm, cn, etc. Representing the vector flux, per centimeter of power conductor for the three conductors, by ψ_a, ψ_b and ψ_c in practical units, and neglecting the capacitance of the power line to ground, the vector potential differences between the three phases of the power line are (Eq. (232))

$$\left.\begin{aligned} E_{ab} &= 2\lambda\left(\psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} + \psi_c ln\frac{D_2}{D_3}\right) \\ E_{bc} &= 2\lambda\left(\psi_a ln\frac{D_3}{D_1} + \psi_b ln\frac{D_2}{r} - \psi_c ln\frac{D_2}{r}\right) \\ E_{ca} &= 2\lambda\left(\psi_a ln\frac{r}{D_3} + \psi_b ln\frac{D_1}{D_2} + \psi_c ln\frac{D_3}{r}\right) \end{aligned}\right\}\text{vector volts} \qquad (294)$$

Where λ is the conversion factor $10^{-9}v^2$ and v is the velocity of light in centimeters per second, or, approximately, $v = 3 \times 10^{10}$ cm. per second.

The vector line potential differences E_{ab}, E_{bc} and E_{ca} are known. Their magnitudes are measurable and the vectors form an equilateral triangle. In the group of three equations (Eq. (294)) there are two independent equations only, containing three unknowns. It is necessary to have one additional, independent equation in order to solve for the three fluxes.

Balanced Three-phase Voltages.—If the three-phase line is balanced, the vector sum of the three potential differences between line conductors and neutral is zero, as is also the vector sum of the dielectric fluxes. Thus, for a balanced line,

$$\psi_a + \psi_b + \psi_c = 0$$

is the third required, independent equation. Once the three fluxes are known, the potential difference induced between the conductors m and n by electrostatic induction may be found It is

$$E_{mn} = 2\lambda\left(\psi_a ln\frac{an}{am} + \psi_b ln\frac{bn}{bm} + \psi_c ln\frac{cn}{cm}\right) \text{ vector volts.} \qquad (295)$$

Since the circuit is balanced, the three equal fluxes are 120 time degrees apart and so may be represented as

$$\psi_a = \psi_0(1 + j0)$$
$$\psi_b = -\frac{\psi_0}{2}(1 + j\sqrt{3})$$
$$\psi_c = \frac{\psi_0}{2}(-1 + j\sqrt{3}).$$

When these values are substituted in Eq. (295), a solution of the resulting equation for the amount of the potential difference yields

$$E_{mn} = 2.302\lambda\psi_0\sqrt{\left(\log_{10}\frac{\overline{an}^2 \cdot bm \cdot cm}{\overline{am}^2\ bn \cdot cn}\right)^2 + 3\left(\log_{10}\frac{cn \cdot bm}{cm \cdot bn}\right)^2} \text{ volts} \qquad (296)$$

This is the potential difference impressed across the capacitance of the communication line at its mid-point. The disturbing current produced by it is the charging current of this condenser. When the condenser is charging, current flows from the two ends of one wire toward its middle, and away from the middle of the second wire toward its ends. When the condenser discharges, the reverse action takes place.

Unbalanced, Three-phase Voltages.—The voltages of a three-phase line are unbalanced when the vector sums of the three voltages to neutral is not zero. The line-to-line voltages supplied by the generators, however, are three equal voltages displaced by 120 time degrees from each other. Thus, in Fig. 48, the three-line-to-line voltages of an unbalanced line are represented by the vectors AB, BC and CA. The three unbalanced voltages to neutral or ground potential are OA, OB and OC. If the voltages were balanced the three potential differences to neutral would be $O'A$, $O'B$ and $O'C$, and the neutral O would be at O'. In the vector diagram however, it is seen that

$$\begin{aligned} OO' + O'A &= OA \\ OO' + O'B &= OB \\ OO' + O'C &= OC \end{aligned}$$

and thus the three unbalanced voltages are equivalent to the three balanced voltages plus three times the potential difference between ground and the neutral of the balanced system. The potential difference OO' is the *residual voltage* of each conductor, while the *residual voltage of the system* is three times this value, and is in phase with OO'.

The potential differences induced in the communication circuit may be estimated either directly or by splitting the three line potential differences to neutral into their residual and balanced components. If the vector potential differences OA, OB and OC are known, the residual voltage may be found.

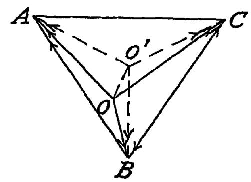


FIG 48.—Residual voltages in unbalanced three-phase lines.

Considering the three line conductors of Fig. 48 and their images (not shown), and using the method of images of Chap. IV, the known vector potential differences to neutral in terms of dielectric fluxes are

$$E_{Oa} = \frac{1}{2}\left[2\lambda\left(\psi_a ln\left(\frac{aa'}{r}\right)^2 + \psi_b ln\left(\frac{ba'}{ba}\right)^2 + \psi_c ln\left(\frac{ca'}{ca}\right)^2\right)\right]$$

$$\left.\begin{aligned} &= 2\lambda\left(\psi_a ln\frac{aa'}{r} + \psi_b ln\frac{ba'}{ba} + \psi_c ln\frac{ca'}{ca}\right) \\ E_{Ob} &= 2\lambda\left(\psi_a ln\frac{ab'}{ab} + \psi_b ln\frac{bb'}{r} + \psi_c ln\frac{cb'}{cb}\right) \\ E_{Oc} &= 2\lambda\left(\psi_a ln\frac{ac'}{ac} + \psi_b ln\frac{bc'}{bc} + \psi_0 ln\frac{cc'}{r}\right) \end{aligned}\right\} \quad (297)$$

where aa', ba' and ca, etc., are the distances between the conductors designated by the letters A and A', B and A', C and A, etc., respectively.

Since the line is unbalanced, the vector sum of the fluxes is not zero. The separate fluxes may, however, be found by solving the above three equations simultaneously. Once the fluxes are known, the induced potential difference between the two wires m and n of the communication circuit may be set down by evaluating the expression

$$E = 2\lambda\psi \int_{x_1}^{x_2} \frac{dx}{x}$$

for each power conductor and its image. The fluxes ψ_a, ψ_b and ψ_c are taken as positive, and are respectively equal to the negatives of the fluxes ψ'_a, ψ'_b and ψ'_c. The values x_1 and x_2 are the distances from the power conductors to the near and far wires of the communication circuit, respectively. Thus, the induced voltage in circuit mn is

$$E_{mn} = 2\lambda\left(\psi_a ln\frac{an \cdot a'm}{am \cdot a'n} + \psi_b ln\frac{bn \cdot b'm}{bm \cdot b'n} + \psi_c ln\frac{cn \cdot c'm}{cm \cdot c'n}\right) \text{ vector volts.} \quad (298)$$

Instead of proceeding as above one may deal with the residual and balanced components of line voltage and their induced potential differences separately. The induced potential difference due to the balanced components may be calculated from Eq. (295), while that due to the residual is

$$\text{Residual } E_{mn} = 2\lambda\psi_R ln\frac{Rn \cdot R'm}{Rm \cdot R'n} \text{ vector volts} \quad (299)$$

where

$$\psi_R = \text{the complex residual flux per centimeter} = \psi_a + \psi_b + \psi_c.$$

R is the point at the center of mass of the conductor configuration triangle, and R' is a point midway between R and its image.

Causes of Current and Voltage Unbalance.—Two general types of networks are in use; namely, (a) insulated systems in which no ground connections are made between neutral and ground at any point, and (b) grounded-neutral systems in which the generator neutrals as well as the neutrals at the load centers are grounded.

In isolated systems perhaps the principal sources of trouble are unbalanced capacitances and leakances between the different phases and ground. The former may largely be corrected by suitable transpositions, while the latter is chiefly a question of careful construction and maintenance. The unbalanced capacitance is less for lines having flat horizontal spacings than for those with vertical or triangular arrangements of conductors.

In grounded-neutral systems unequal load impedances from line to neutral will cause unbalanced components of current to flow in the neutral or ground return. Third harmonics and their odd multiples are set up by the cyclic variation of the core flux in transformers, the percentage of the harmonic being dependent, to a considerable extent, upon the degree of saturation of the core. In a grounded-neutral system using star-to-star transformer connections, these harmonics will appear as unbalanced currents flowing in the neutral connection, the currents due to the three phases being additive in the neutral return. The use of delta connection on the secondary side provides a short-circuit path for these harmonics and greatly reduces them. When a resistance or a reactance is inserted in the connection between neutral and ground, currents flowing to ground will cause voltage unbalance.

Harmonics.—The current and voltage waves of power circuits are always more or less distorted. The positive and negative loops of succeeding cycles, however, are alike in shape. If the negative loop be rotated about the zero axis and then moved over to match the positive loop, the two loops are found to be identical in shape. Owing to symmetry of construction, all waves generated by rotating, electrical machinery have this type of symmetry. Composite waves of this type are in fact made up of a number of sine waves of various frequencies, consisting of fundamental and odd integral multiples of the fundamental frequency. Harmonics may be introduced by generators or large synchronous motors on account of pulsating, air-gap reluctance in slotted armatures, by transformers due to the cyclic variation of core reluctances, by certain types of load, such as electric arcs and rectifiers, and by the formation of corona at the peaks of the voltage waves.

From the standpoint of interference, the harmonics, even though of relatively small magnitudes, are of greater importance than the fundamental. The disturbing effect set up in a telephone

circuit depends upon the amount of the induced voltage and upon the frequency of the induced current. Since the voltage itself is proportional to the frequency, within the approximate range between say 200 and 800 cycles, the disturbing effect is proportional to the square of the frequency. Frequencies which fall within the range of the average voice frequency produce far greater disturbances per unit of induced voltage than do those lying without or near one end of the voice-frequency band.

The voice-frequency band approximately covers the range from 200 cycles per second to 2,000 cycles per second. Near its middle, or in the neighborhood of 800 cycles per second, lies what is usually taken as the average voice frequency. At approximately this frequency, the telephone receiver is most sensitive The standard fundamental frequency of 50 or 60 cycles, therefore, is ordinarily of little importance in producing disturbances in telephone lines, while in telegraph lines this frequency may be the source of interference troubles.

PROBLEMS

1. The three equal conductors of a 60-kv , three-phase, 60-cycle power line are each 0 533 in in diameter The conductors are supported at the vertices of an equilateral triangle whose base is parallel to the earth's surface Beginning with the uppermost conductor and reading counter-clockwise, the conductors are a, c, b, the conductor a lying on the center line of the supporting structure The conductors are 6 ft apart between centers, and the plane cb is 40 ft above ground Fifty feet to the right of the center line of the power circuit is the center line of a horizontal telephone loop mn, running parallel to the power circuit. The telephone wire m is 30 ft. above ground, and the width of the loop is 2 ft. The balanced voltages to neutral of the power circuit are

$$\left.\begin{aligned} E_{0a} &= 34,600(1 + j0) \\ E_{0b} &= -17,300(1 + j\sqrt{3}) \\ E_{0c} &= 17,300(-1 + j\sqrt{3}) \end{aligned}\right\} \text{vector volts}$$

and the balanced currents are each equal to 175 amp and lag the above voltages by 30 time degrees, respectively. Calculate the value of (a) the electromagnetically-induced voltage, and (b) the electrostatically-induced voltage in the telephone loop per mile of untransposed parallel.

2 A 110-kv., 60-cycle, three-phase power line is built of three conductors, each of 0.631-in. diameter. They are strung 11 ft. apart in a horizontal plane, 40 ft. above ground, the middle conductor lying on the center line of the support. The center line of a horizontal telephone loop is 60 ft. to the right of the center line of the power line, and its conductors m and n are 2 ft apart. Reading from left to right in the diagram representing the arrangement of lines described above, the power conductors are a, b and c,

and the telephone conductors are m and n. The line-to-line voltage E_{ab} of the power line is

$$E_{ab} = 110{,}000 + j0$$

and the voltages to neutral are

$$\begin{aligned} E_{0a} &= 34{,}900 + j55{,}000 \\ E_{0b} &= 75{,}100 + j55{,}000 \\ E_{0c} &= 20{,}100 - j40{,}300. \end{aligned}$$

Assuming the earth's potential as zero, what is the electrostatically-induced voltage E_{mn}?

CHAPTER VII

SHORT-LINE CALCULATIONS

The structure of a transmission line is such that its electrical properties per unit length of line are practically constant. These properties are represented by the symbols r, g, L and C, and are called the line constants. The magnitudes of these constants depend only upon the conductor size, conductor material, and conductor arrangements in space.

Because the so-called constants of a given transmission line do not vary, the transfer of energy over the line follows laws which may readily be expressed in simple mathematical form when the current and voltage are sinusoidal. When automatic regulators are used to maintain constant voltage at either one or both ends of the line and at all loads, the performance calculations are still further simplified.

The exact equations of the transmission line involving the correct assumption of uniformly distributed, line capacitance, while simple in form, involve considerable calculation In order to simplify the task of making transmission-line calculations, where such simplification is permissible without sacrificing too much in accuracy, two expedients have been resorted to.

1. The use of the first one or two terms only of the converging infinite series which are the equivalents of the hyperbolic functions used in the exact equations. This simplifies the operations considerably, but the error increases as the number of terms used in each series diminishes.

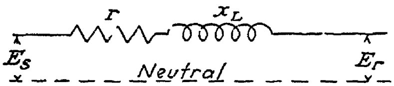
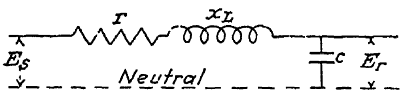
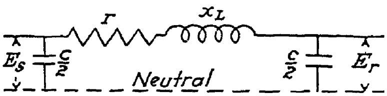
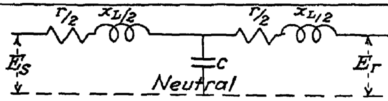
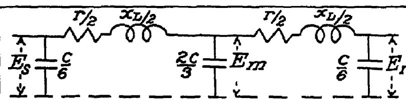
2. The substitution of hypothetical circuits having lumped capacitance or capacitances (or no capacitance at all), which perform very nearly like the actual transmission line. Under this classification various kinds of substitute circuits are possible.

The method described under (1) above will be discussed in the chapter dealing with the exact equations of the line. Under (2) a number of different approximations of various degrees of complexity are used, depending principally upon the length of the line under consideration, and the degree of accuracy desired.

These substitute circuits differ from each other chiefly in the methods employed to approximate the effect of the line capacitance. In lines under 30 miles in length the capacitance of the line is often entirely neglected in making line calculations.

A list of the substitute circuits used in line calculations is given in Table 8.

TABLE 8 —TYPES OF SUBSTITUTE CIRCUITS FOR APPROXIMATE CALCULATIONS

METHOD	CIRCUIT
Impedance	
Load-end Condenser	
Nominal π	
Nominal T	
Dr Steinmetz' Three Condenser Method	
H B. Dwights' K Formulas	See Transmission Line Formulas by H. B Dwight

In addition to the above, graphical and semigraphical methods are available.

Simple Impedance Circuits.—In the simplest approximate circuit the line is assumed to have no capacitance. The circuit then reduces to a series circuit containing only resistance and inductive reactance. This approximately equivalent line will have a greater line drop, and, usually, a somewhat larger generator current and poorer power factor than the actual line. For 60-

cycle lines under 30 miles long delivering fair sized loads, however, the error is usually negligible. Even for 60-cycle lines up to 50 miles the error will probably usually not exceed 0.6 per cent. Thus, for lines up to perhaps 25 or 30 miles, there is little practical objection to the use of this approximately equivalent circuit.

In Fig. 49 is represented one leg of a three-phase line, over which one-third of the power delivered to the line is assumed to be transmitted. It is further assumed that the receiver-end conditions are known, that is, the receiver load, voltage, current and power factor. The problem is to find the corresponding quantities at the generator or input end, the line regulation, losses, etc. Quantities carrying the subscripts r refer to the receiver end, while those carrying the subscript s refer to the generator or supply end For quantities which are the same at both ends the

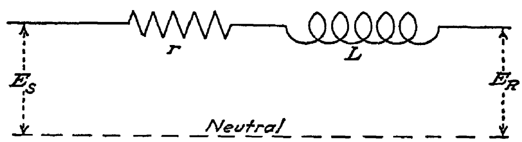


FIG 49 —Diagram of simple impedance circuit

subscripts are omitted. The supply-end voltage is assumed to be controlled by varying the generator excitation to whatever extent is necessary to keep the receiver voltage constant in value. The receiver voltage is used as the vector of reference. Since the conductance of the leakage path and the condensance of the line are both negligibly small, the supply and receiver currents are one and the same. Thus, assuming that the receiver current may be either leading or lagging, the line current is

$$I = I_1 \pm jI_2 \text{ vector amp.}$$

and the receiver power factor angle is

$$\theta_r = \pm \tan^{-1}\frac{I_2}{I_1}.$$

The line impedance is

$$Z = r + jx \text{ vector ohms}$$

where

r = the total resistance of one conductor

and

x = the total inductive reactance of one conductor.

The impedance drop in the line is

$$\begin{aligned} ZI &= (r + jx)(I_1 \pm jI_2) \\ &= rI_1 \mp xI_2 + j(xI_1 \pm rI_2) \qquad (300) \\ &= I_1(r + jx) + I_2(\mp x \pm jr) \text{ vector volts.} \qquad (301) \end{aligned}$$

The supply-end voltage is the receiver voltage plus the impedance drop, or

$$\begin{aligned} E_s &= E_r + ZI \\ &= E_r + rI_1 \mp xI_2 + j(xI_1 \pm rI_2) \qquad (302) \\ &= {}_sE_1 + j{}_sE_2 \text{ vector volts} \end{aligned}$$

where ${}_sE_1$ and ${}_sE_2$ are the real and the quadrature components of the supply voltage, respectively. The angle of the supply voltage referred to the receiver voltage is

$$ {}_e\theta_s = \tan^{-1} \frac{{}_sE_2}{{}_sE_1} \qquad (303)$$

and the supply power factor angle is

$$\begin{aligned} \theta_s &= {}_e\theta_s - (\pm\theta_r) \\ &= {}_e\theta_s \mp \theta_r. \qquad (304) \end{aligned}$$

In terms of absolute values, the supply voltage is

$$\begin{aligned} E_s &= \sqrt{{}_sE_1^2 + {}_sE_2^2} \\ &= \sqrt{(E_r + rI_1 + xI_2)^2 + (xI_1 \pm rI_2)^2} \text{ volts.} \qquad (305) \end{aligned}$$

Note that, wherever the double signs appear, the upper one is the condition for leading and the lower one for lagging current.

The percentage of line regulation found by substituting full-load current values in Eq. (305) is

$$\text{per cent regulation} = \frac{100(E_s - E_r)}{E_r}. \qquad (306)$$

The supply output is

$$Kw_s = E_sI \cos \theta_s \qquad (307)$$

the receiver input is

$$Kw_r = E_rI \cos \theta_r \qquad (308)$$

and the line loss is

$$\left.\begin{aligned} \text{Loss} &= E_sI \cos \theta_s - E_rI \cos \theta_r \\ \text{or} \qquad \text{Loss} &= \frac{I^2r}{1{,}000} \text{ kw.} \end{aligned}\right\} \qquad (309)$$

Vector Diagram for Impedance Circuit.—The vector diagram for this case is conveniently built up by adding the impedance drop of the line, as given by Eq. (301), to the receiver voltage to obtain the supply voltage. Thus, in Fig. 50, assuming a lagging current having components I_1 and $-jI_2$, the four component drops are laid off as indicated in the solid-line diagram. The first term of the right-hand member of Eq. (301) gives the drop due to the in-phase component of the current, I_1, resulting in the right triangle of which ZI_1 is the hypotenuse, while the second term yields the drop due to the quadrature component of the line current, as shown in the right triangle of which ZI_2 is the hypotenuse. The vector sum of ZI_1 and ZI_2 is the total impedance drop ZI. It is to be noted that the two right triangles above

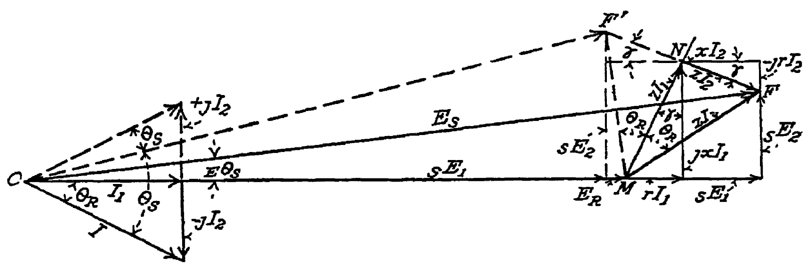


FIG. 50.—Vector diagram for the simple impedance circuit

referred to are similar since their acute angles are each equal to the angle γ. That is,

$$\gamma = \tan^{-1}\frac{rI_1}{xI_1} = \tan^{-1}\frac{rI_2}{xI_2} = \tan^{-1}\frac{r}{x}.$$

The line ZI_2 is therefore perpendicular to ZI_1. Furthermore, in the triangle MNF, angle $NMF = \theta_r$, the receiver power factor angle, for

$$\begin{aligned}\angle NMF &= \tan^{-1}\frac{ZI_2}{ZI_1}\\ &= \theta_r.\end{aligned}$$

The diagram may readily be extended to represent any power factor by simply including a series of lines to represent different values of θ_r. The effect, upon the receiver voltage, of introducing leading or lagging components of current, is clearly shown in the diagram. For given values of supply voltage, increasing lagging quadrature currents depress the receiver voltage, while increasing leading quadrature currents cause it to rise. This

changing effect is due to the angle at which the line impedance drop subtracts vectorially from the generator voltage to produce the receiver voltage.

Perrine-Baum Diagram.—This diagram is useful as an aid in visualizing the performance of a short line, and serves at the same time as a means of solving short-line problems. Graphical solutions taken from carefully drawn diagrams will be of sufficient accuracy for practical purposes. It is always desirable, however, to make at least one purely mathematical check calculation, when a graphical method is used.

The Perrine-Baum diagram follows at once by somewhat generalizing Fig. 50. The diagram is illustrated in Fig. 51. To draw the diagram, the procedure is as follows: Choose a suitable

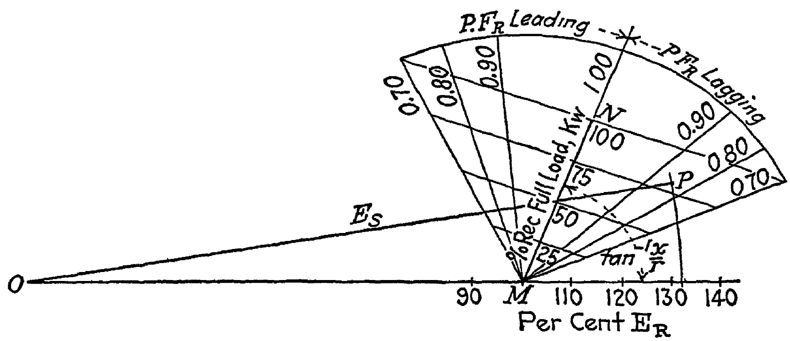


FIG. 51 —The Perrine-Baum diagram

scale, and lay off OM along the X-axis equal to the receiver voltage E_r. Draw MN, making an angle $\theta = \tan^{-1}\frac{x}{r}$ with OM (x and r are the reactance and resistance of one line conductor, respectively). The length of MN to the voltage scale is the product of the full-load value of I_1 and the line impedance; that is,

$$MN = I_1\sqrt{r^2 + x^2} = I_1 z.$$

MN thus represents the impedance drop in the line at full load and unity power factor. By dividing MN into segments proportional to the loads, the impedance drop at unity power factor for any load may be read directly. Draw radial power factor lines to represent various power factors, both leading and lagging, such as MF, MF' etc., of Fig. 50. Lay off a scale of voltages along OM in percentage of the receiver voltage; that is, $OM = E_r = 100$ per cent, etc. This will serve as a measure of the supply voltage

when its vector is swung down to parallelism with OM. The line regulation may then readily be computed.

To illustrate the use of the diagram, suppose the line is loaded to 80 per cent of full-load kilowatts at a power factor of 85 per cent, current lagging. The end of the E_s vector is then found at P, and the supply voltage is seen to be 132 per cent of the receiver voltage.

Mershon Diagram.—Referring to Fig. 49, if the current be used as the axis of reference, the impedance drop in the line is then

$$ZI = I(r + jx) \text{ vector volts}$$

where I is the load current as before. The supply voltage is the sum of the receiver voltage and the impedance drop or

$$E_s = E_r + rI + jxI \text{ vector volts.} \tag{310}$$

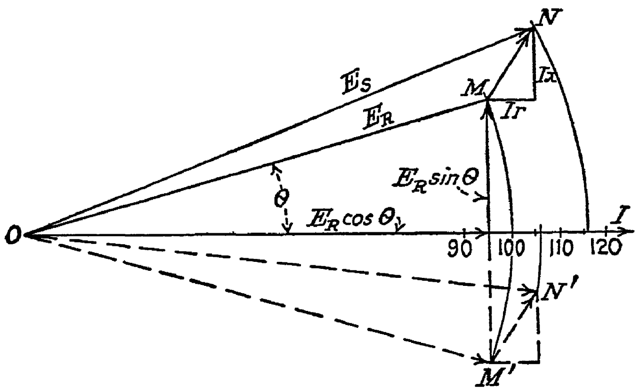


FIG 52.—The Mershon diagram

The receiver voltage will lag behind the load current for leading power factors, and *vice versa* for lagging power factors; or, in general,

$$E_r = E_r(\cos\theta_r \mp j\sin\theta_r) \text{ vector volts} \tag{311}$$

where the upper sign refers to leading currents, and $\cos\theta_r$ is the power factor of the receiver load. Combining Eqs. (310) and (311), the generator voltage is

$$E_s = E_r\cos\theta_r + Ir \mp j(E_r\sin\theta_r + Ix) \text{ vector volts.} \tag{312}$$

The vector diagram, of voltages corresponding to Eq. (312), is shown in Fig. (52). The solid-line diagram is for lagging current while the dotted-line diagram represents leading current.

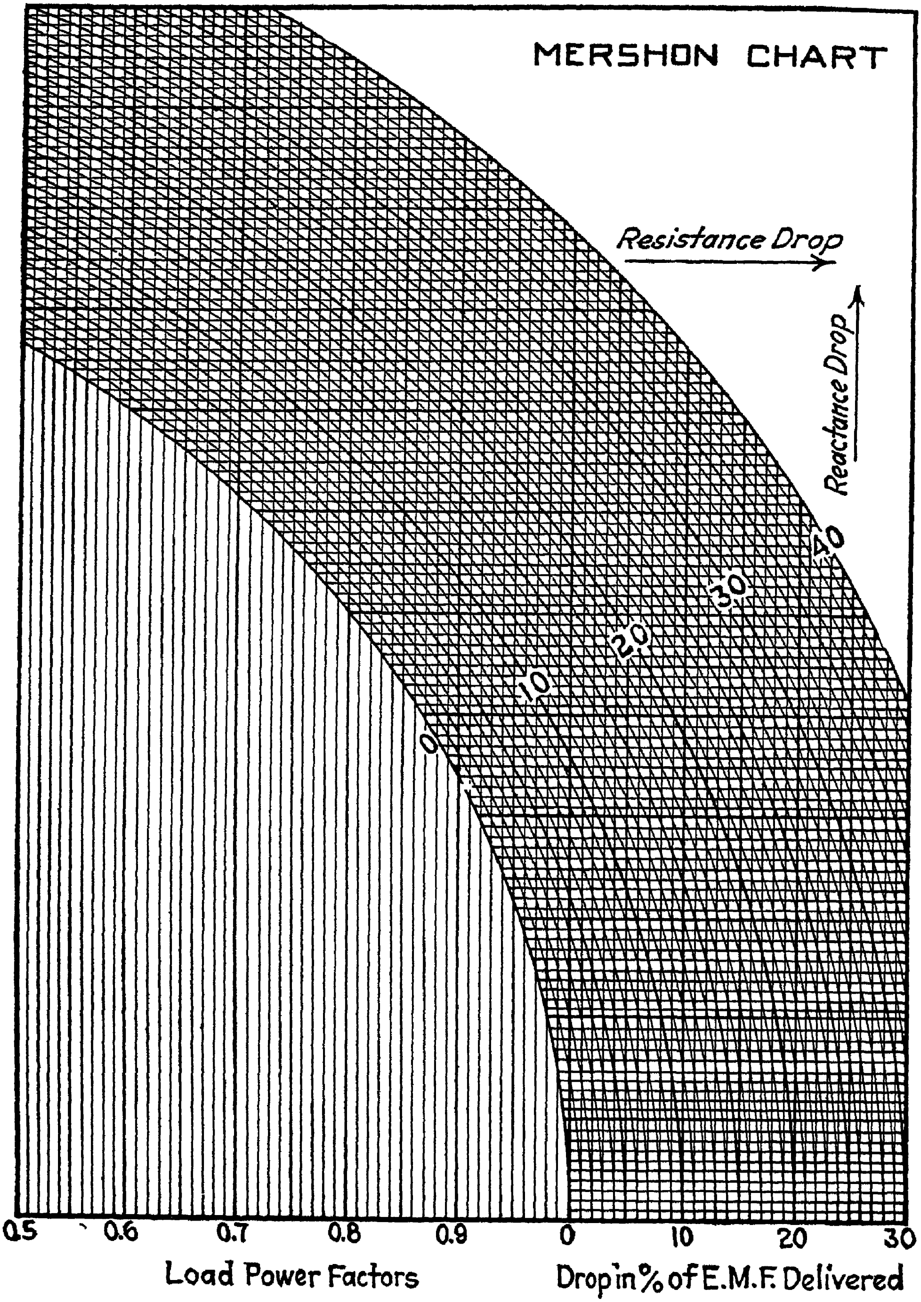


FIG. 53.—The Mershon chart.

It is apparent that if the receiver voltage remains constant in size, the point M will trace the arc of a circle as the power factor angle changes. Whatever the position of E_r, however, the impedance-drop line MN always has the same slope, and therefore adds vectorially to E_r at varying angles, to produce the supply voltage E_s. By expressing all components of voltage in percentage of E_r, the latter being 100 per cent, laying the percentage scale off on the X-axis as indicated, and drawing in a series of circles at 1 or 2 per cent intervals, the basis for the Mershon chart becomes apparent. This chart simply provides a ready means of adding the per cent impedance drop vector MN to E_r at any angle within the scope of the chart, and, by following down the arc of the circle passing through N, provides a means of reading off the scale on the X-axis the corresponding line drop in per cent of the receiver voltage. The Mershon chart is reproduced in Fig. 53.

Effect of Line Capacitance.—While the effect of line capacitance is negligible in 60-cycle lines up to 20 or 30 miles in length, it is a very important factor in the operation of very long lines, and, in general, must at least be approximated in the calculations for all lines except the short ones mentioned above. Since the various approximate solutions differ largely in the method used to approximate the capacitance effect, it is important to have a clear understanding of what happens in the circuit.

Capacitance, like resistance and inductance, is uniformly distributed along the line. Each unit length of line has a certain definite value of C, depending upon the size of conductors and their interaxial spacings Each unit therefore requires a reactive component of leading current to charge the capacitance associated therewith. This current is a constant amount per unit length of line per volt of potential difference, and leads the potential difference between line conductors at the particular point considered by 90°. Since the vector-line voltage varies from point to point along the line, both in size and in angular position, however, it is apparent that the quadrature component, of current per line element, also varies in size and phase from point to point. Furthermore, since the line current is the sum of the load current and the aggregate of all quadrature currents required to charge the elements of the line lying between the chosen point and the receiver, the current in the line varies from point to point, and, likewise, the drop in the series impedance of the line per unit of length is a variable. Leading current flowing in the inductive

reactance of the line causes the voltage to rise towards the end of the line. The line capacitance thus has the effect of decreasing the line drop for a given load current (Figs. 50 and 52).

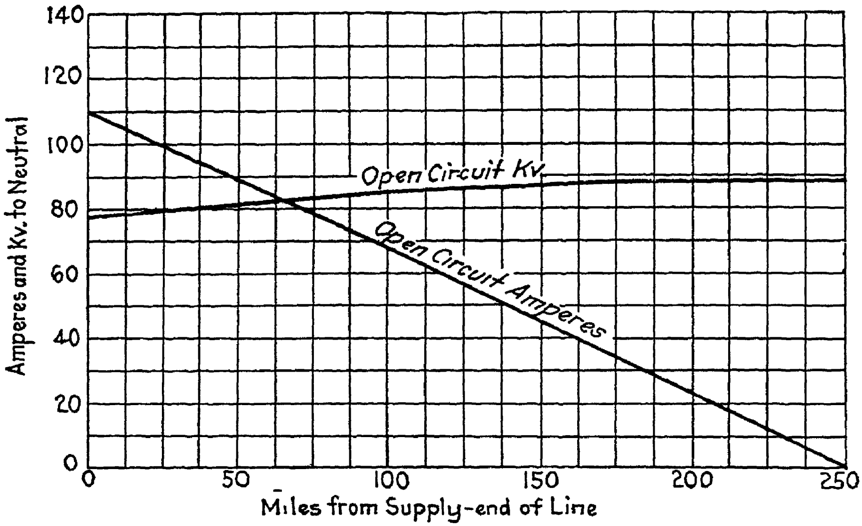


FIG. 54.—Open-circuit current and voltage of a 250-mile transmission line

Charging Current.—Consider a transmission line which is open at the receiver end and is subjected to a potential difference at the supply end, of a value such that the receiver-end voltage is

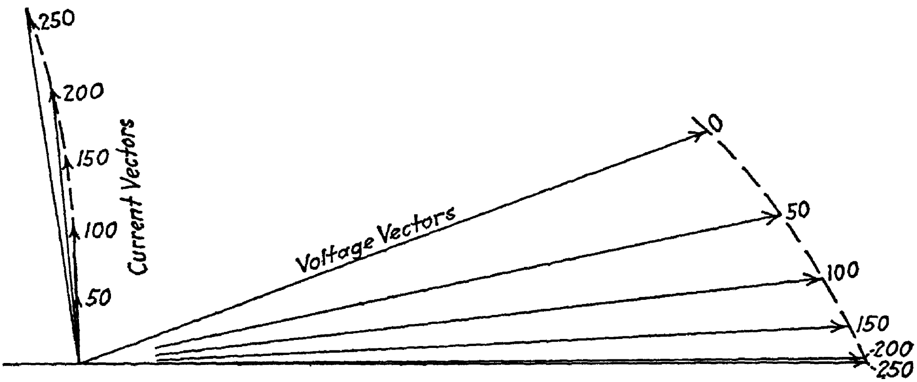


FIG 55.—Open-circuit current and voltage vectors of a 250-mile transmission line.

normal. The current in such a line is zero at the receiver and increases towards the supply end, where it has its maximum value. The latter value is called the *charging current* of the line. The

voltage is maximum at the receiver end and decreases towards the supply end. In a long line the former may be considerably higher than the latter. For example, in the 200-mile line of Chap. XVII, with normal supply voltage impressed, the no-load receiver-end voltage rises to a value of 22.9 per cent above the supply-end voltage. The no-load charging current for the same line is 29.1 per cent of the full-load supply current or 88 1 amp., for normal receiver voltage. The variations in the size, of no-load current and voltage vectors along a 250-mile line for normal receiver-end voltage, are illustrated in the curves of Fig. 54. In Fig. 55 the corresponding polar diagrams are drawn.

Load-end Condenser Method.—The roughest approximation in which the line capacitance is considered, assumes a circuit like the second one of Table 8. Here the capacitance of one line conductor is assumed to be placed across the line in parallel with the load at the receiver end. The charging current is thus the product of the receiver voltage to neutral and the admittance of the capacity to neutral, and is assumed to be constant throughout the line, and leading the receiver voltage by 90°. The line current is the vector sum of the load current and the line-charging current, and the line drop is the product of this current and the series impedance of the line. The method of calculation, therefore, differs only slightly from that already given for the simple impedance circuit, and will not be developed in detail. The use of this method results in a generator voltage which is too low by about the same amount as it is too high when the impedance method is used. An average of the values obtained by the two methods is more nearly correct than either value. The impedance method is uncompensated for the rise in voltage towards the receiver end, due to the charging current, while the load-end condenser method overcompensates for this effect.

Nominal π Line.—This is the third circuit of Table 8. Here only one-half the line capacitance is assumed to be put in parallel with the load, the remaining one-half being put from line to neutral at the generator end. The method of calculation is identical with that of the single-load-end condenser, except that

(*a*) Only one-half of the line-charging current of the former is added to the load current in the latter.

(*b*) The supply current is the vector sum of the line current and the charging current of the supply-end condenser. The results secured by this method are considerably more accurate than those resulting from the use of the load-end condenser circuit.

Nominal T Line.—An inspection of Figs. 54 and 55 suggests the use of the approximation employed in this circuit. From these figures it is observed that the open-circuit current increases at a fairly uniform rate along the line. No great error will result in a short line if it be assumed as a constant amount per mile of line. Similarly, the drop in voltage along the line is fairly uniform, and the value of the potential difference at the middle of the line is nearly equal to the average value. If a line having only resistance and inductive reactance be assumed, and if a lumped capacitance, equal to the capacitance of one line conductor to neutral, be placed across the line at its mid-point, the charging current flowing in it will be approximately equal to the actual charging current, and the drop in the impedance of one-half the line due to this current, will be approximately equal to the drop in the actual line. This type of approximate circuit is called the "nominal T" line.

The circuit for this line is given below in Fig. 56.

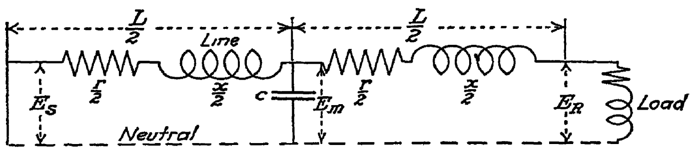


FIG. 56.—The nominal T line.

Representing the load current by

$$I_r = I_1 - jI_2 \text{ vector amp.}$$

the potential difference across the lumped capacitance is

$$\begin{aligned} E_m &= E_r + \frac{1}{2}(I_1 - jI_2)(r + jx) \\ &= E_r + \frac{1}{2}(rI_1 + xI_2) + j\,\frac{1}{2}(xI_1 - rI_2) \qquad (313) \\ &= {}_mE_1 + jmE_2 \text{ vector volts} \end{aligned}$$

where r and x are the resistance and reactance of one line conductor and ${}_mE_1$ and ${}_mE_2$ are the real and the quadrature components of the potential difference E_m, respectively.

The size of the vector E_m is

$$E_m = \sqrt{{}_mE_1^2 + {}_mE_2^2}. \qquad (314)$$

The current flowing in the line capacitance is

$$I_c = jb_cE_m \text{ vector amp.}$$

where $b_c = 2\pi fC$, the susceptance of one line conductor to neutral. Thus, from Eq (313),

$$I_c = jb_c\left[E_r + \frac{1}{2}(rI_1 + xI_2) + j\frac{1}{2}(xI_1 - rI_2)\right]$$
$$= b_c\left\{\frac{1}{2}(rI_2 - xI_1) + j\left[E_r + \frac{1}{2}(rI_1 + xI_2)\right]\right\}\text{vector amp.} \quad (315)$$

and

$$I_s = I_r + I_c \text{ vector amp.}$$
$$= I_1 - jI_2 + I_c$$
$$= I_1 + b_c\left[\frac{1}{2}(rI_2 - xI_1)\right] + j\left\{b_c\left[E_r + \frac{1}{2}(rI_1 + xI_2)\right] - I_2\right\} \quad (316)$$
$$= {}_sI_1 \pm j{}_sI_2 \text{ vector amp.} \quad (317)$$

where ${}_sI_1$ and ${}_sI_2$ represent the real and quadrature components, respectively, of the supply current, as found from Eq. (316).

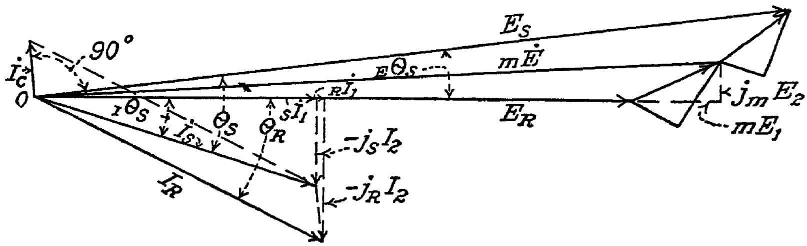


FIG 57 —Vector diagram for the nominal T line

It is to be noted that the sign of ${}_sI_2$ of Eq. (317) may be either positive or negative, depending upon the relative magnitudes of $-I_2$ and the remainder of the j-component. Thus the second term may be either $+j{}_sI_2$ or $-j{}_sI_2$. The double sign here does not mean that it will have two values.

The angle of the supply current is

$$_I\theta_s = \tan^{-1}\frac{{}_sI_2}{{}_sI_1}. \quad (318)$$

The supply voltage is

$$E_s = E_m + \frac{1}{2}ZI_s$$
$$= E_m + \frac{1}{2}(r + jx)({}_sI_1 \pm j{}_sI_2)$$
$$= {}_sE_1 \pm j{}_sE_2 \text{ vector volts} \quad (319)$$

and its length is $E_s = \sqrt{{}_sE_1^2 + {}_sE_2^2}$ volts. (320)

The angle of the supply-voltage vector is

$$ {}_e\theta_s = \pm \tan^{-1} \frac{{}_sE_2}{{}_sE_1} \tag{321}$$

and the supply power factor becomes

$$\cos \theta_s = \cos ({}_e\theta_s \pm {}_i\theta_s). \tag{322}$$

The line loss may now be computed. It is

$$\begin{aligned} \Sigma I^2 r &= E_s I_s \cos \theta_s - E_r I_r \cos \theta_r \\ &= \frac{r}{2}(I_s^2 + I_r^2) \text{ watts.} \end{aligned} \tag{323}$$

The vector diagram for the nominal T line is shown in Fig. 57.

The vector diagram of Fig. 50 may also be modified to compensate approximately for the rise in voltage due to the charging current of the nominal T line. The receiver voltage is assumed constant. The charging current is calculated, and is assumed to lead the receiver voltage by 90° at all loads. (This assumption is not strictly true for any load, but is very nearly so.) Referred to the receiver voltage, the charging current is jI_c, and the potential drop in one-half of the line due to this current is

$$j\frac{ZI_c}{2} = \frac{1}{2}(-xI_c + jrI_c) \text{ vector volts.}$$

This is approximately equal to the true rise in voltage in the entire line due to the actual charging current. The vector

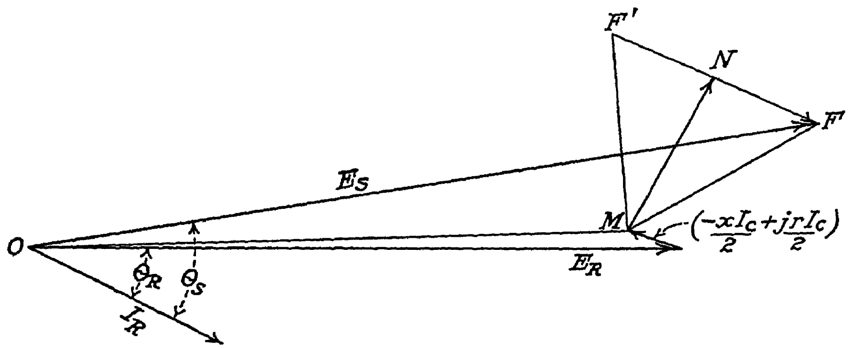


FIG. 58.—Diagram of voltages including the effect of charging current.

diagram of Fig. 50 may therefore be modified to include the effect of the charging current by adding the above value to the receiver voltage of Fig. 50, to locate the point M in Fig. 58. This

addition assumes that the charging current leads E_r by 90°, which is of course, not strictly true. The method of constructing the remainder of the diagram is the same as for Fig. 50.

Dr. Steinmetz' Split-condenser Method.—This method assumes the line capacitance to be divided into three parts, placed as shown in Fig. 59, $\frac{1}{6}C$ at each end of the line and $\frac{2}{3}C$ at the middle of the line, where C is the total capacitance of one conductor to neutral. The calculations are carried out step fashion in much the same way as for the nominal T method. Using the same notation as before, except that the currents in the two halves of the line are given the subscripts a and b in conformity with the notation in the figure, the charging current of load-end condenser is

$$I_c = j\frac{2\pi fCE_r}{6} = j_rI_c \text{ vector amp.}$$

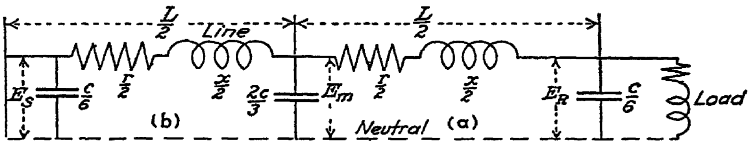


FIG. 59.—The Steinmetz split-condenser circuit.

For lagging power factors the load current is

$$I_r = I_1 - jI_2$$

whence

$$I_a = I_1 - jI_2 + j_rI_c$$
$$= I_1 - j(I_2 - {}_rI_c) \text{ vector amp.}$$

$$E_m = E_r + \frac{ZI_a}{2}$$
$$= E_r + \frac{1}{2}(r + jx)[I_1 - j(I_2 - {}_rI_c)]$$
$$= E_r + \frac{1}{2}[rI_1 + x(I_2 - {}_rI_c)] + j\frac{1}{2}[(xI_1 - r(I_2 - {}_rI_c)] \quad (324)$$
$$= {}_mE_1 + j{}_mE_2 \text{ vector volts}$$

and, in amount,

$$E_m = \sqrt{{}_mE_1^2 + {}_mE_2^2} \text{ volts absolute.} \quad (325)$$

The charging current of the middle condenser is

$$
\begin{aligned}
{}_mI_c &= jb_mE_m = j\frac{4\pi fC}{3}({}_mE_1 + j\,{}_mE_2) \\
&= \frac{4\pi fC}{3}(-{}_mE_2 + j\,{}_mE_1) \qquad (326)\\
&= -{}_{mc}I_1 + j\,{}_{mc}I_2 \text{ vector amp.}
\end{aligned}
$$

Then

$$
\begin{aligned}
I_b &= I_a + {}_mI_c \\
&= I_1 - {}_{mc}I_1 - j(I_2 - {}_{rc}I_1 - {}_{mc}I_2) \qquad (327)\\
&= {}_bI_1 \pm j\,{}_bI_2 \text{ vector amp.} \qquad (328)
\end{aligned}
$$

The sign of I_b may be either positive or negative, depending upon the relative values of quantities within the parenthesis, hence the double sign in Eq. (328). The potential drop in the impedance of the remaining one-half of the line is

$$
\begin{aligned}
\frac{ZI_b}{2} &= \frac{1}{2}(r + jx)({}_bI_1 - j\,{}_bI_2) \\
&= \frac{1}{2}(r_bI_1 + x_bI_2) + j\frac{1}{2}(x_bI_1 - r_bI_2) \text{ vector volts} \qquad (329)
\end{aligned}
$$

and

$$
\begin{aligned}
E_s &= E_m + \frac{ZI_b}{2} \\
&= E_r + \frac{r}{2}(I_1 + {}_bI_1) + \frac{x}{2}(I_2 - {}_rI_c + {}_bI_2) \\
&\qquad + j\left[\frac{x}{2}(I_1 + {}_bI_1) - \frac{r}{2}(I_2 - {}_rI_c + {}_bI_2)\right] \qquad (330)\\
&= {}_sE_1 + j\,{}_sE_2 \text{ vector volts.} \qquad (331)
\end{aligned}
$$

Also

$$E_s = \sqrt{{}_sE_1^2 + {}_sE_2^2} \text{ volts absolute} \qquad (332)$$

and

$${}_e\theta_s = \tan^{-1}\frac{{}_sE_2}{{}_sE_1}. \qquad (333)$$

The charging current of the generator-end condenser is

$$
\begin{aligned}
{}_sI_c &= jb_sE_s \\
&= j\frac{2\pi fC}{6}({}_sE_1 + j\,{}_sE_2) \\
&= \frac{2\pi fC}{6}(-{}_sE_2 + j\,{}_sE_1) \\
&= -{}_{sc}I_1 + j\,{}_{sc}I_2 \text{ vector amp.} \qquad (334)
\end{aligned}
$$

and the generator current is

$$
\begin{aligned}
I_c &= I_b + {}_sI_c \\
&= {}_bI_1 - j_bI_2 - {}_{sc}I_1 + j_{sc}I_2 \\
&= {}_bI_1 - {}_{sc}I_1 + j({}_{sc}I_2 - {}_bI_2) && (335) \\
&= {}_sI_1 - j_sI_2 \text{ vector amp.} && (336)
\end{aligned}
$$

or

$$I_s = \sqrt{{}_sI_1^2 + {}_sI_2^2} \text{ volts absolute.} \qquad (337)$$

The angle of I_s is

$$\pm {}_i\theta_s = \tan^{-1} \frac{{}_sI_2}{{}_sI_1} \qquad (338)$$

and the generator power factor is

$$\cos \theta_s = \cos ({}_e\theta_s \pm {}_i\theta_s). \qquad (339)$$

In Eq. (339) the sign to be chosen for the second term is determined by the sign of ${}_i\theta_s$. The generator output and the line loss readily follow from the above relations.

This approximate method has the advantage of being sufficiently accurate in preliminary calculations for lines of any length yet built, but it is rather long and cumbersome.

Impedance of Line Transformers.—In the previous discussions of line calculations the impedance of raising and lowering transformers has been omitted. The transformer resistances and reactances, however, are usually not negligible. They may be combined with the line constants to give approximately correct results by simply adding the equivalent values, of resistance and reactance of transformers per line to neutral, to the corresponding constants already found for the line alone. This method of including the impedance offered by the transformers neglects the effect of the exciting current of the transformers, and is therefore not strictly correct. A method in which the admittance of the transformer exciting circuit is also taken into account will be found in a later chapter.

The constants of transformers are obtainable from the manufacturer, and are usually expressed as a percentage of the transformer voltage. They may be calculated in terms of either the high-side or the low-side voltage. For transformers of the types and sizes used in transmission line work, the resistance drop will probably be in the neighborhood of 0.6 per cent, and the reactance drop about 7.5 or 8 per cent.

Consider a three-phase line whose kilowatt power output per leg is

$$P_r = E_r I_r \cos\theta_r \times 10^{-3} \tag{340}$$

where

E_r = receiver-end line voltage to neutral

and

I_r = receiver-end line current.

Then, if

d_x = transformer reactance drop in per cent of the high-side voltage

d_r = transformer resistance drop in percentage of the high-side voltage

x = equivalent reactance per leg of lowering transformers

r = equivalent resistance per leg of lowering transformers

we have,

$$\frac{d_x}{100} = \frac{xI_r}{E_r}$$

or

$$x = \frac{E_r d_x}{100 I_r}. \tag{341}$$

Similarly

$$r = \frac{E_r d_r}{100 I_r}. \tag{342}$$

Substituting the value of I_r from Eq. (340) into Eqs. (341) and (342),

$$x = \frac{10^{-5} E_r^2 \cos\theta_r \cdot d_x}{P_r} \text{ ohms} \tag{343}$$

$$r = \frac{10^{-5} E_r^2 \cos\theta_r \cdot d_r}{P_r} \text{ ohms.} \tag{344}$$

The constants for the raising transformers at the supply end are found in a similar manner. The sum of the lowering and raising transformer reactances is then added to the line reactance to give the equivalent reactance for the entire line including transformers. The equivalent line resistance is obtained in an exactly similar manner. The solution of a line problem may then be carried forward in the manner already indicated for the various approximate methods.

PROBLEMS

1. A 35-mile, 60-cycle, three-phase line has three 0000 solid copper conductors of 0.533-in. diameter, supported at the vertices of an equilateral

triangle that is 6 ft. on a side. The resistance of each conductor is 0.26 ohm per mile. Consider the capacitance of the line as concentrated at the middle. If the receiver voltage is held constant at 50 kv., what is the supply voltage: (*a*) at no load? (*b*) at full load of 16,000 kw and 85 per cent lagging power factor? (*c*) What is the line loss at full load?

2 Recalculate (*a*), (*b*) and (*c*) of Problem 1 neglecting charging current.

3. A three-phase, 60-cycle line, 105 miles long, has the following constants per mile:

$$r = 0\ 130 \text{ ohm}$$
$$x = 0\ 805 \text{ ohm}$$
$$b = 5\ 12 \times 10^{-6} \text{ mho.}$$

The full-load receiver input is 60,000 kw. at 85 per cent lagging power factor. The receiver voltage is held constant at 154 kv. For full-load receiver input, calculate by Dr Steinmetz' split-condenser method the following supply-end quantities. (*a*) voltage, (*b*) current, (*c*) power factor. What is the per cent line loss at full load? Using the receiver voltage as the axis of reference, draw vector diagrams showing the various component voltages and currents that combine to make the supply voltage and the supply current.

4. Recalculate Problem 3, assuming the total line capacitance to be concentrated in the middle of the line Assuming the split-condenser method to give correct results, calculate the per cent error in each of the quantities called for in Problem 3.

CHAPTER VIII

LONG-LINE EQUATIONS. EXACT SOLUTION

LONG TRANSMISSION LINES

The Electric Circuit.—When dealing with the electric circuits of three-phase transmission lines it is most convenient to consider only one leg of the circuit, that is, one line conductor whose potential above ground is the potential E_n to neutral. The circuit under consideration may then be pictured as in Fig. 60. It consists of an infinite number of elemental circuits in series, parallel grouping. Since the line conductors are of uniform section and, on the average, are spaced throughout the line at a constant distance from the other conductors of the line, each elemental

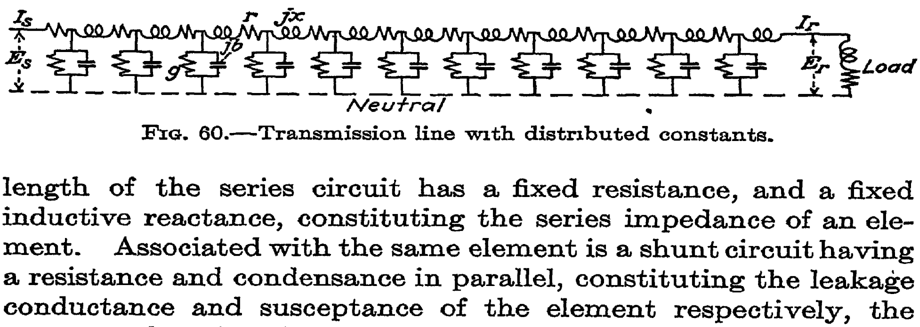


FIG. 60.—Transmission line with distributed constants.

length of the series circuit has a fixed resistance, and a fixed inductive reactance, constituting the series impedance of an element. Associated with the same element is a shunt circuit having a resistance and condensance in parallel, constituting the leakage conductance and susceptance of the element respectively, the two together forming the admittance of the leakage path. Between any two adjacent elements there is a drop of potential in the series circuit, due to the line current flowing in the impedance of the element, and so the potential to neutral at the beginning of the first element differs from that at the beginning of the second element, both in magnitude and in phase position. Likewise, the current in the first element differs from that in the second element by an amount equal to the current which flows in the admittance of the shunt circuit between them. In general, the current in the line therefore also varies from point to point both in size and in phase position.

One exception to the above statements is to be noted. This occurs when the voltage impressed between line conductors, and the current flowing in them, are of such values per unit length of line that the energy, stored in the magnetic field of the line during one-half cycle, is exactly equal to the energy released from the dielectric field during the same half cycle. Under these conditions, if the power factor of the receiver circuit is unity, no reactive power flows in any part of the line, the current in all parts of the line is the same, and the line drop is pure rI drop, in phase with the current at all points.

If one were to draw a diagram showing the vector values of voltages and currents along the line for given values of receiver load current and voltage, using a line conductor itself as one axis of a three-dimensional diagram, the result would be a picture somewhat as shown in Fig. 61. The surfaces formed by

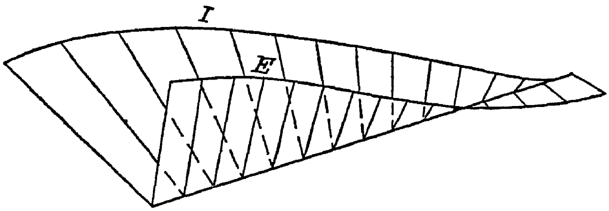


Fig. 61.—Perspective drawing of current and voltage vectors along a transmission line

the current and voltage vectors are warped surfaces; the surface elements at any point in the line are the current and voltage vectors, and the angle between them is the power factor angle at the point in question.

The difference between the vector impedance drops of adjacent elements, or between the vector currents in adjacent elements of a line, is small. In short-line calculations the approximations made usually consist in assuming that the current flowing is the same, and that the impedance drop is constant, for all elements. In medium-length lines the line is sometimes divided into two equal sections with the above quantities assumed constant for each section but having different values for the two sections, as illustrated by the discussion of the previous chapter. In long lines, however, these differences existing cannot be neglected if accurate results are desired.

Approximate Methods.—While several of the approximate methods of Chap. VII yield results sufficiently accurate for

CHAPTER VIII

LONG-LINE EQUATIONS. EXACT SOLUTION

LONG TRANSMISSION LINES

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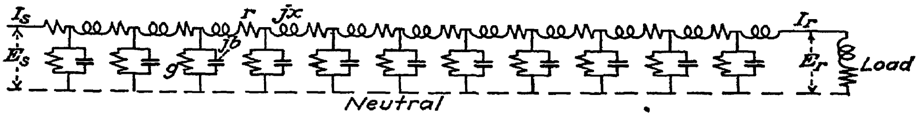

FIG. 60.—Transmission line with distributed constants.

length of the series circuit has a fixed resistance, and a fixed inductive reactance, constituting the series impedance of an element. Associated with the same element is a shunt circuit having a resistance and condensance in parallel, constituting the leakage conductance and susceptance of the element respectively, the two together forming the admittance of the leakage path. Between any two adjacent elements there is a drop of potential in the series circuit, due to the line current flowing in the impedance of the element, and so the potential to neutral at the beginning of the first element differs from that at the beginning of the second element, both in magnitude and in phase position. Likewise, the current in the first element differs from that in the second element by an amount equal to the current which flows in the admittance of the shunt circuit between them. In general, the current in the line therefore also varies from point to point both in size and in phase position.

and the potential difference from line to neutral at $l + dl$ miles from the receiver is

$$E + dE = E_1 + (I_1r_1 + I_2x_1)dl + j[E_2 + (I_1x_1 \mp I_2r_1)dl]$$
vector volts. (346)

The current in the leakage path of the element dl is

$$\begin{aligned} dI &= (E_1 + jE_2)(g_1 + jb_1)dl \\ &= (E_1g_1 - E_2b_1)dl + j(E_1b_1 + E_2g_1)dl \text{ vector amp.} \quad (347) \end{aligned}$$

The positive sign is used with dI because the charging current increases in the positive direction of l.

The current in the line, at a distance $l + dl$ miles from the receiver, is

$$I + dI = I_1 + (E_1g_1 - E_2b_1)dl - j[I_2 - (E_1b_1 + E_2g_1)dl].$$

The vector diagram showing these voltage and current increments of an element of line is shown in Fig. 62.

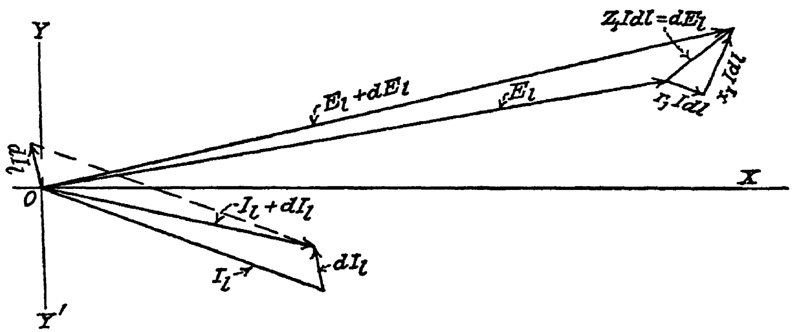

FIG. 62.—Voltage and current diagrams for element dl of line.

Using the more general form of notation, the increments, of vector-voltage rise in the element of the series circuit, is

$$dE = IZ_1dl$$

or

$$\frac{dE}{dl} = IZ_1 \quad (348)$$

and the current, leaking to neutral from length dl of line, is

$$dI = EY_1dl$$

or

$$\frac{dI}{dl} = EY_1. \quad (349)$$

Equations (348) and (349) are the fundamental differential equations of the circuit under consideration. Their solution will give the desired values of I and E at any point in the line.

Solution of Equations.—Differentiating Eq (348) with respect to l,

$$\frac{d^2E}{dl^2} = Z_1\frac{dI}{dl} \tag{350}$$

Differentiating Eq. (349) with respect to l,

$$\frac{d^2I}{dl^2} = Y_1\frac{dE}{dl}. \tag{351}$$

Substituting Eq. (349) in Eq (350),

$$\frac{d^2E}{dl^2} = Y_1Z_1E. \tag{352}$$

Substituting Eq (348) in Eq. (351),

$$\frac{d^2I}{dl^2} = Y_1Z_1I. \tag{353}$$

It will be observed that Eqs (352) and (353) are alike in form, and must therefore have the same general solution. Whatsoever differences appear in the values of E and I will result from constants of integration, i $e.$, from assumed terminal conditions.

Multiplying both sides of Eq. (353) by $\frac{dI}{dl}$,

$$\frac{dI}{dl}\cdot\frac{d^2I}{dl^2} = Y_1Z_1I\ \frac{dI}{dl}. \tag{354}$$

Integrating Eq. (354),

$$\frac{1}{2}\left(\frac{dI}{dl}\right)^2 = \frac{Y_1Z_1I^2}{2} + \frac{c^2}{2} \tag{355}$$

the last term representing the constant at integration.

Let

$$c = Y_1Z_1c_1^2.$$

Then, from Eq. (355),

$$\frac{dI}{dl} = \sqrt{Y_1Z_1}\sqrt{I^2 + c_1^2} \tag{356}$$

or

$$\frac{dI}{\sqrt{I^2 + c_1^2}} = \sqrt{Y_1Z_1}\cdot dl. \tag{357}$$

Equation (357) may be integrated by making the following substitutions:

Let

$$I = c_1 \sinh u$$

Then

$$dI = c_1 \cosh u \cdot du$$

and

$$\sqrt{I^2 + c_1^2} = c_1\sqrt{\sinh^2 u + 1} = c_1 \cosh u.$$

Making these substitutions in Eq (357) and solving for u,

$$\int du = \sqrt{Y_1Z_1}\int dl + c_2$$

or

$$u = \sqrt{Y_1Z_1}l + c_2 \tag{358}$$

and

$$I = c_1 \sinh\ (\sqrt{Y_1Z_1}l + c_2). \tag{359}$$

Differentiating Eq. (359),

$$\frac{dI}{dl} = \sqrt{Y_1Z_1}c_1 \cosh\ (\sqrt{Y_1Z_1}l + c_2). \tag{360}$$

Substituting Eq (349) in Eq (360) and solving for E,

$$E = \sqrt{\frac{Z_1}{Y_1}}c_1 \cosh\ (\sqrt{Y_1Z_1}l + c_2). \tag{361}$$

Equations (359) and (361) are the desired equations for I and E in which it remains to determine the constants c_1 and c_2 from known terminal conditions.

In power transmission problems the current and voltage at the receiver end of the line are usually known or may readily be estimated. If l be taken as positive in the direction from receiver to generator, the assumed known terminal conditions (for $l = 0$) are

$$E = E_r \text{ the receiver vector voltage}$$

and

$$I = I_r \text{ the receiver vector current.}$$

Expanding Eq. (359),

$$I = c_1\,(\sinh \sqrt{Y_1Z_1}l \cdot \cosh c_2 + \cosh \sqrt{Y_1Z_1}l \cdot \sinh c_2) \tag{362}$$

or, since

$$I = I_r \text{ for } l = 0$$
$$I_r = c_1 \sinh c_2$$

and

$$\sinh c_2 = \frac{I_r}{c_1}. \tag{363}$$

Expanding Eq. (361),

$$E = c_1\sqrt{\frac{Z_1}{Y_1}}(\cosh\sqrt{Y_1Z_1}l \cosh c_2 + \sinh\sqrt{Y_1Z_1}l \sinh c_2) \tag{364}$$

and since

$$E = E_r \text{ when } l = 0$$

$$E_r = c_1\sqrt{\frac{Z_1}{Y_1}} \cosh c_2$$

or

$$\cosh c_2 = \frac{E_r}{c_1\sqrt{\frac{Z_1}{Y_1}}}. \tag{365}$$

Substituting Eqs. (363) and (365) in Eqs. (362) and (364),

$$E = E_r \cosh\sqrt{Y_1Z_1}l + I_r\sqrt{\frac{Z_1}{Y_1}} \sinh\sqrt{Y_1Z_1}l \tag{366}$$

$$I = I_r \cosh\sqrt{Y_1Z_1}l + E_r\sqrt{\frac{Y_1}{Z_1}} \sinh\sqrt{Y_1Z_1}l. \tag{367}$$

Since E_r and I_r, respectively, represent the voltage to neutral and the current in a line conductor at the receiver or load end of the line, E and I of Eqs. (366) and (367) are the corresponding quantities at any point on the line a distance l from the receiver end. If the assumed known conditions be taken at the supply end of the line as E_s and I_s, instead of at the receiver end, then, since the distance along the line is now measured in the direction of the flow of energy, that is, from generator towards receiver, l becomes negative. Remembering that

$$\cosh\sqrt{Y_1Z_1}(-l) = \cosh\sqrt{Y_1Z_1}l$$

and

$$\sinh\sqrt{Y_1Z_1}(-l) = -\sinh\sqrt{Y_1Z_1}l$$

it is apparent that, with the new set of terminal conditions, the general equations of the line become

$$E = E_s \cosh\sqrt{Y_1Z_1}l - I_s\sqrt{\frac{Z_1}{Y_1}} \sinh\sqrt{Y_1Z_1}l \tag{368}$$

$$I = I_s \cosh\sqrt{Y_1Z_1}l - E_s\sqrt{\frac{Y_1}{Z_1}} \sinh\sqrt{Y_1Z_1}l. \tag{369}$$

These equations yield the current and voltage of the line at any point distant l units from the supply end and in terms of the current and voltage at that end.

If the impedance and admittance of the entire line be used instead of the values per unit of length, the substitutions given below are required.

$Z = Z_1 l =$ the total series impedance on one conductor.
$Y = Y_1 l =$ the total shunted admittance of one conductor.

When these substitutions are made, Eqs. (366) and (367) become

$$E_s = E_r \cosh \sqrt{YZ} + I_r \sqrt{\frac{Z}{Y}} \sinh \sqrt{YZ} \tag{370}$$

$$I_s = I_r \cosh \sqrt{YZ} + E_r \sqrt{\frac{Y}{Z}} \sinh \sqrt{YZ}. \tag{371}$$

Equations (370) and (371) are the general equations for the entire line in terms of the assumed known values of E_r and I_r. The equations are not only exact, but are very convenient to use in the solution of transmission-line problems.

The notation of Eq. (370) and (371) may be somewhat simplified by writing

$$E_s = E_r A + I_r B \tag{372}$$
$$I_s = I_r A + E_r C \tag{373}$$

where the complex numbers A, B and C are derived or auxiliary line constants.

Still another variation in the form of these equations is obtained by expressing the various complex numbers involved by their respective cartesian components. Using the vector E_r as the zero or reference vector, the equations become

$$\begin{aligned} E_s &= E_1 + jE_2 \\ &= E_r(a_1 + ja_2) + ({}_rI_1 - j_rI_2)(b_1 + jb_2) \end{aligned} \tag{374}$$
$$\begin{aligned} I_s &= {}_sI_1 + j_sI_2 \\ &= ({}_rI_1 - j_rI_2)(a_1 + ja_2) + E_r(c_1 + jc_2) \end{aligned} \tag{375}$$

where

$$A = a_1 + ja_2$$
$$B = b_1 + jb_2$$
$$C = c_1 + jc_2$$

are the complex, derived line constants, and

$$\begin{aligned} E_s &= {}_sE_1 + j_sE_2 \\ I_s &= {}_sI_1 + j_sI_2 \\ E_r &= E_r + j0 \\ I_r &= {}_rI_1 - j_rI_2. \end{aligned}$$

When it comes to the study of line performance, Eqs. (374) and (375) are probably the most helpful of any. With their use one may readily draw the loci of current and voltage vectors for any loads and power factors. The method will be discussed in detail in another chapter.

Derived or Auxiliary Line Constants.—The symbols used in the simplified form of the above equations are interpreted as follows:

$$\begin{aligned} A &= \cosh \sqrt{YZ} \\ &= a_1 + ja_2, \text{ a complex number} \qquad (376) \\ B &= \sqrt{\frac{Z}{Y}} \sinh \sqrt{YZ} \\ &= Z_0 \sinh \sqrt{YZ} \\ &= b_1 + jb_2, \text{ a complex number} \qquad (377) \\ C &= \sqrt{\frac{Y}{Z}} \sinh \sqrt{YZ} \\ &= Y_0 \sinh \sqrt{YZ} \\ &= c_1 + jc_2, \text{ a complex number.} \qquad (378) \end{aligned}$$

It may be noted also that

$$Z_0 = \sqrt{\frac{Z}{Y}}, \text{ a complex number, called the } \textit{surge impedance} \text{ of the line,} \qquad (379)$$

$$\begin{aligned} Y_0 &= \sqrt{\frac{Y}{Z}} \\ &= \frac{1}{Z_0}, \text{ a complex number called the } \textit{surge admittance} \text{ of the line,} \qquad (380) \end{aligned}$$

and

$$\begin{aligned} \sqrt{YZ} &= \theta \\ &= \alpha + j\beta, \text{ a complex number called the } \textit{complex angle} \text{ of the line.} \qquad (381) \end{aligned}$$

The complex numbers A, B and C are called the *derived* or *auxiliary constants* of the line. They are hyperbolic functions of the complex line angle $\sqrt{YZ}$ subtended by the line, and depend only upon the *fundamental* line constants r, g, L and C and the frequency. Since the values of r, g, C and L are directly proportional to the length of the line, the derived constants also are a function of the length. The surge impedance and the surge admittance are both independent of the length of the line. They depend only upon the values of resistance, inductance, conductance and capacitance *per unit length* of line.

The Complex Line Angle $\theta = \sqrt{YZ}$.—Since, in general, Z and Y are complex numbers, it follows that the same is true of the line angle θ. Also, since $\sqrt{YZ} = \sqrt{Y_1l \cdot Z_1l}$, the unit line angle $\theta_1 = \theta \div l = \sqrt{Y_1Z_1}$. That is, associated with each unit length of a smooth line having uniformly distributed constants r_1, g_1, L_1 and C_1 per unit length of line, and operating at a fixed frequency, is a unit complex angle $\theta_1 = \sqrt{Y_1Z_1}$. It consists of a real part or hyperbolic component, and an imaginary part or circular component Custom has established the use of the symbols α_1 and β_1 respectively, to represent these components. Hence the unit line angle may be written as

$$\theta_1 = \sqrt{Y_1Z_1} = \alpha_1 + j\beta_1. \tag{382}$$

Multiplying this equation through by the length of line and writing $\alpha = \alpha_1l$ and $\beta = \beta_1l$, the complex angle of the whole line is

$$\theta = \sqrt{YZ} = \alpha + j\beta. \tag{383}$$

The real part α is measured in hyperbolic radians, while the imaginary part β is measured in circular radians. β is reduced to degrees by multiplying by 57 296, the number of degrees in a radian of circular measure

Surge Impedance.—The surge impedance of a smooth, alternating-current line is defined by Eq. (379) as

$$\begin{aligned} Z_0 &= \sqrt{\frac{Z}{Y}} \\ &= \sqrt{\frac{r + jx}{g + jb}}. \end{aligned} \tag{384}$$

The operation indicated by the right-hand member of Eq. (384) is most readily performed when the polar form of expression is employed. This notation yields the expression,

$$Z_0 = \sqrt{\frac{z/\theta_z}{y/\theta_y}}$$
$$= \sqrt{\left(\frac{r^2 + x^2}{g^2 + b^2}\right)^{\frac{1}{2}} \Big/ \tfrac{1}{2}(\theta_z - \theta_y)}$$
$$= \sqrt[4]{\frac{r^2 + x^2}{g^2 + b^2}} \Big/ \gamma \text{ vector ohms} \tag{385}$$
$$= z_0 (\cos \gamma + j \sin \gamma) \text{ vector ohms} \tag{386}$$

where

$$\theta_z = \tan^{-1} \frac{x}{r}$$
$$\theta_y = \tan^{-1} \frac{b}{g}$$

and

$$\gamma = \frac{1}{2}(\theta_z - \theta_y) \text{ radians.} \tag{387}$$

It is apparent from Eq. (379) that the surge impedance also is independent of the length of line, but is determined solely by the values of r, L, g and C per unit length of line, and by the frequency. For 60-cycle lines of usual construction, the size of Z_0 usually lies between 350 and 400 ohms. The angle γ, called the *characteristic phase angle* of the line, has a small, negative value of from 3 to 6° for very heavy, high-voltage lines, while for lighter lines and lower voltages it may be as high as 15 to 20°.

The surge impedance of a smooth line is equal to the impedance which would be measured at the generator end of the line if its length approached infinity. For, since by Eq. (368),

$$E_r = E_s \cosh \sqrt{YZ} - I_s Z_0 \sinh \sqrt{YZ}$$

the receiver voltage approaches zero as the length of line approaches infinity. Hence, for very large values of l,

$$E_s \cosh \sqrt{YZ} = I_s Z_0 \sinh \sqrt{YZ}$$

and

$$\frac{E_s}{I_s} = Z_0 \tanh \sqrt{YZ}$$
$$= Z_0$$
$$= z_0 (\cos \gamma + j \sin \gamma)$$

since

$$\tanh \infty = 1$$

The surge impedance of a line may be calculated from measurements of the line impedance taken at the supply end, (*a*) with the receiver end open, and (*b*) with the receiver end short-circuited. It is found to be the geometric mean of the supply end impedances calculated from the above two sets of readings. The proof of this relation follows.

When the receiver circuit is open the receiver current is zero, and hence, for this condition, by Eqs. (366) and (367),

$$E_s = E_r \cosh \sqrt{YZ}$$
$$I_s = E_r Y_0 \sinh \sqrt{YZ}$$

or

$$Z_{Oc} = \frac{E_s}{I_s} = Z_0 \coth \sqrt{YZ}.$$

Similarly, when the line is short-circuited at the receiver, the receiver voltage is zero and

$$E_s = I_r Z_0 \sinh \sqrt{YZ}$$
$$I_s = I_r \cosh \sqrt{YZ}$$

or

$$Z_{sc} = \frac{E_s}{I_s} = Z_0 \tanh \sqrt{YZ}$$

whence

$$Z_0^2 = Z_{Oc} \times Z_{sc}$$

or

$$Z_0 = \sqrt{Z_{Oc} \times Z_{sc}}.$$

It should also be noted that if the leakage conductance and the resistance are negligibly small as compared with the susceptance and the inductive reactance, respectively, the numerical value of the surge impedance reduces to

$$Z_0 = \sqrt{\frac{L}{C}} \text{ vector ohms}$$

and its angle approaches zero. For heavy power lines using conductors of large areas, the resistance per mile is very low, while the values of the reactance and susceptance per mile are practically independent of the size of the conductor employed; for a given frequency they are nearly constant for all lines, since the spacing of the conductors is approximately proportional to the line voltage.

Surge Admittance.—The surge admittance is the reciprocal of the surge impedance. It is

$$Y_0 = \frac{1}{Z_0} = \sqrt[4]{\frac{g^2 + b^2}{r^2 + x^2}}\underline{/-\gamma} \text{ vector mho}$$
$$= y_0(\cos \gamma - j \sin \gamma) \text{ vector mho.} \qquad (388)$$

Its size is the reciprocal of the size and its slope the negative of the slope of the surge impedance, *i.e.*,

$$y_0 = \frac{1}{z_0}$$

and

$$\theta_{y0} = -\theta_{z0}.$$

Solution for α and β.—In terms of the line constants r, x, g and b, the values of α and β may be obtained as follows:[1]

$$\alpha + j\beta = \sqrt{YZ}$$
$$= \sqrt{(r + jx)(g + jb)}. \qquad (389)$$

Squaring Eq. (389),

$$\alpha^2 + 2j\alpha\beta - \beta^2 = gr - bx + j(br + gx)$$

or

$$\alpha^2 - \beta^2 = gr - bx \qquad (390)$$

and

$$2\alpha\beta = br + gx. \qquad (391)$$

Squaring Eqs. (390) and (391) and adding,

$$(\alpha^2 + \beta^2)^2 = (r^2 + x^2)(g^2 + b^2)$$
$$= z^2y^2$$

or

$$\alpha^2 + \beta^2 = zy. \qquad (392)$$

Adding Eqs. (390) and (392) and solving

$$\alpha = \sqrt{\frac{1}{2}(zy + gr - bx)} \text{ hyperbolic radians.} \qquad (393)$$

Subtracting Eq. (390) from Eq. (392) and solving,

$$\beta = \sqrt{\frac{1}{2}(zy - gr + bx)} \text{ circular radians.} \qquad (394)$$

If the insulation of the line is so good that the leakage conductance may be taken equal to zero, the condition which usually

[1] STIENMETZ, C. P., "Transient Phenomena," 3d ed., p. 292.

prevails in high-tension power lines, then $g = 0$ and $y = b$. On the basis of this assumption, the equations for α and β reduce to

$$\alpha = \sqrt{\frac{b}{2}(z - x)} \tag{395}$$

$$\beta = \sqrt{\frac{b}{2}(z + x)}. \tag{396}$$

Since, for power lines, z and x are usually nearly equal in size, the square root of their difference becomes a somewhat uncertain quantity. Accordingly, it is not always convenient or desirable to use Eqs. (395) and (396) to evaluate α and β In such cases it is preferable to obtain these quantities directly by performing the operation indicated by $\sqrt{YZ}$. The polar expression for the angle will serve best for this purpose.

Representing the impedance and admittance angles of the line by

$$\theta_z = \tan^{-1}\frac{x}{r}$$

and

$$\theta_y = \tan^{-1}\frac{b}{g}$$

and putting

$$\delta = \frac{1}{2}(\theta_z + \theta_y)$$

the polar expression for the line angle becomes

$$\alpha + j\beta = \sqrt{YZ} = \sqrt[4]{(r^2 + x^2)(g^2 + b^2)}\,\underline{/\tfrac{1}{2}(\theta_z + \theta_y)} \tag{397}$$

$$= \theta\underline{/\delta} \text{ vector radians} \tag{398}$$

where, in size,

$$\theta = \sqrt[4]{(r^2 + x^2)(g^2 + b^2)}. \tag{399}$$

Upon again reducing Eq. (398) to rectangular coordinates and equating reals and imaginaries separately, the values of α and β are found to be

$$\left.\begin{aligned} \alpha &= \theta \cos \delta \text{ hyperbolic radians} \\ \beta &= \theta \sin \delta \text{ circular radians} \end{aligned}\right\} \tag{400}$$

Again, when g is approximately equal to zero, the values of θ and δ are, approximately,

$$\theta = \sqrt{bz} \tag{401}$$

and

$$\delta = \frac{\pi}{4} + \frac{\theta_z}{2}. \tag{402}$$

Variation of α and β with Line Constants and Frequency.—It is of interest to consider the manner in which α and β vary with the constants r, L, g and C and the frequency. If the frequency is zero, then $(x = \omega L) = 0 = (b = \omega C)$,

whence

$$\begin{aligned} \theta &= \sqrt{(r + j0)(g + j0)} \\ &= \sqrt{rg} \underline{/\delta} \end{aligned}$$

where

$$\begin{aligned} \delta &= \frac{1}{2}\left(\tan^{-1}\frac{x}{r} + \tan^{-1}\frac{b}{g}\right). \\ &= 0 \end{aligned}$$

Therefore,

$$\alpha = \sqrt{rg} \text{ hyperbolic radians}$$

and

$$\beta = 0 \text{ circular radians.}$$

This condition prevails in a leaky, direct-current line. In such lines the voltage and current are attenuated in size but not in phase.

On the other hand, if $r = 0 = g$ and the frequency is not zero,

$$\begin{aligned} \theta &= \sqrt{j^2 bx} \\ &= j2\pi f\sqrt{LC} \end{aligned}$$

and

$$\delta = \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned} \alpha &= 0 \text{ hyperbolic radians} \\ \beta &= 2\pi f\sqrt{LC} \text{ circular radians.} \end{aligned}$$

While the resistance in a power line is never negligible, and α, therefore, is not zero, yet as compared with β, α is very small, and β is approximately equal to the value given.

For example, consider a 60-cycle line built of 500,000-cir. mil copper cable, for which the constants per mile are $r_1 = 0.022$, $x_1 = 0.82$, $b_1 = 5.2 \times 10^{-6}$, $g = 0$ (approximately). The resistance, being very small as compared with the reactance, will not materially affect the value of β_1. The latter is

$$\begin{aligned}\beta_1 &= \sqrt{x_1 b_1}\\ &= 0.00206 \text{ radian per mile}\\ &= 0.118^\circ \text{ per mile}\end{aligned}$$

This means that the component voltage and current vectors swing through an angle of 0 118° per mile of line.

Forms of Expressions for the Constants A, B and C.—The equations of the long line may be put into a number of different forms, depending upon the expressions which are used for the constants A, B and C and upon the grouping of terms. One useful form is illustrated in Eqs. (366) and (367). Other useful forms are considered below.

a. The Hyperbolic Forms.—In Eqs (366) and (367) the constants are expressed in terms of hyperbolic functions of the complex angle of the line They are

$$\left.\begin{aligned}A &= \cosh\sqrt{YZ} = \cosh(\alpha + j\beta)\\ B &= Z_0 \sinh\sqrt{YZ} = Z_0 \sinh(\alpha + j\beta)\\ C &= Y_0 \sinh\sqrt{YZ} = Y_0 \sinh(\alpha + j\beta)\end{aligned}\right\} \qquad (403)$$

If charts[1] and tables[2] of complex hyperbolic numbers are available, the constants may be computed from Eqs. of (403) as they stand. In the absence of such aids, and particularly for precise calculations, more convenient forms of expression to use are those in which only circular and hyperbolic functions of real numbers appear. These are obtained by the expansion of the sinh and cosh of the complex variable. They are

$$\left.\begin{aligned}A &= \cosh\alpha\cos\beta + j\sinh\alpha\sin\beta\\ B &= Z_0(\sinh\alpha\cos\beta + j\cosh\alpha\sin\beta\\ C &= Y_0(\sinh\alpha\cos\beta + j\cosh\alpha\sin\beta\end{aligned}\right\} \qquad (404)$$

With the aid of tables of hyperbolic functions and circular functions of real numbers and a computing machine, precise calculations are easily made from these equations.

b. Convergent Series Form with Complex Numbers.—By substituting for the hyperbolic functions of Eqs of (403) the appropriate series in $\sqrt{YZ}$, writing for Z_0 and Y_0 the equivalents $\sqrt{\frac{Z}{Y}}$ and $\sqrt{\frac{Y}{Z}}$, respectively, and simplifying the resulting expressions,

[1] KENNELLY, A. E., "Chart Atlas of Complex Hyperbolic and Circular Functions."

[2] KENNELLY'S "Tables of Complex Numbers."

the complex convergent series forms for A, B and C are obtained. They are

$$\left.\begin{aligned} A &= 1 + \frac{YZ}{2} + \frac{Y^2Z^2}{24} + \frac{Y^3Z^3}{720} + \cdots \\ B &= Z\left(1 + \frac{YZ}{6} + \frac{Y^2Z^2}{120} + \frac{Y^3Z^3}{5{,}040} + \cdots\right) \\ C &= Y\left(1 + \frac{YZ}{6} + \frac{Y^2Z^2}{120} + \frac{Y^3Z^3}{5{,}040} + \cdots\right) \end{aligned}\right\} \tag{405}$$

c. Convergent Series Form with Real Numbers.—Very convenient convergent series forms of expression, using real numbers only, are obtained as follows:

$$\left.\begin{aligned} A &= \cosh(\alpha + j\beta) \\ &= \frac{1}{2}(\epsilon^{\alpha+j\beta} + \epsilon^{-\alpha-j\beta}) \\ &= \frac{1}{2}(\epsilon^{\alpha}\epsilon^{j\beta} + \epsilon^{-\alpha}\epsilon^{-j\beta}) \\ &\text{and similarly,} \\ B &= \frac{Z_0}{2}(\epsilon^{\alpha}\epsilon^{j\beta} - \epsilon^{-\alpha}\epsilon^{-j\beta}) \\ C &= \frac{Y_0}{2}(\epsilon^{\alpha}\epsilon^{j\beta} - \epsilon^{-\alpha}\epsilon^{-j\beta}) \end{aligned}\right\} \tag{406}$$

If, due to the excellence of line insulation, it may be assumed that $g = 0$, then $Y = jb$ and

$$\begin{aligned} Z_0 &= \sqrt{\frac{Z}{Y}} = \sqrt{\frac{YZ}{Y^2}} = \frac{\alpha + j\beta}{jb} \\ &= \frac{\beta - j\alpha}{b} \end{aligned} \tag{407}$$

$$\begin{aligned} Y_0 &= \frac{1}{Z_0} = \frac{b}{\beta - j\alpha} \\ &= \frac{\beta + j\alpha}{z}. \end{aligned} \tag{408}$$

By substituting in Eqs. of (406) the corresponding equivalent series in place of ϵ^{α} and $\epsilon^{-\alpha}$, for Z_0 and Y_0 the expressions of Eqs. (407) and (408), respectively, and for $\epsilon^{j\beta}$ and $\epsilon^{-j\beta}$ the expressions

$$\begin{aligned} \epsilon^{j\beta} &= \cos\beta + j\sin\beta \\ \epsilon^{-j\beta} &= \cos\beta - j\sin\beta \end{aligned}$$

there results, after simplifying, the rapidly converging series,

$$\left.\begin{aligned}
A &= \left(1 + \frac{\alpha^2}{2} + \frac{\alpha^4}{24} + \frac{\alpha^6}{720} + \cdots\right)\cos\beta \\
&\qquad + j\left(\alpha + \frac{\alpha^3}{6} + \frac{\alpha^5}{120} + \frac{\alpha^7}{5{,}040} + \cdots\right)\sin\beta \\
B &= \frac{\beta - j\alpha}{b}\left[\left(\alpha + \frac{\alpha^3}{6} + \frac{\alpha^5}{120} + \frac{\alpha^7}{5{,}040} + \cdots\right)\cos\beta\right. \\
&\qquad \left. + j\left(1 + \frac{\alpha^2}{2} + \frac{\alpha^4}{24} + \frac{\alpha^6}{720} + \cdots\right)\sin\beta\right] \\
C &= \frac{\beta + j\alpha}{z}\left[\left(\alpha + \frac{\alpha^3}{6} + \frac{\alpha^5}{120} + \frac{\alpha^7}{5{,}040} + \cdots\right)\cos\beta\right. \\
&\qquad \left. + j\left(1 + \frac{\alpha^2}{2} + \frac{\alpha^4}{24} + \frac{\alpha^6}{720} + \cdots\right)\sin\beta\right]
\end{aligned}\right\} \quad (409)$$

The errors introduced by omitting powers of α above the second, or at most the third, will usually be negligible. Hence, for practical calculations the constants are readily evaluated from Eq. (409) if desired

Generalized Line Constants of Equivalent Networks.[1]—When networks consist of various combinations of series and parallel groupings of their constituent elements, it is frequently desirable to combine the elements into a simple *equivalent network* having only the usual generalized constants. When a single line alone is considered, the supply-end voltage and current are given by the equations,

$$E_s = E_rA + I_rB$$
$$I_s = I_rA + E_rC.$$

Frequently, however, a transmission network may consist of two or more lines in parallel; or a line having certain characteristics may be connected in series with another having different characteristics. Transformers form an essential part of a transmission network, and their impedances must be considered in evaluating potential drops, etc , between receiver and supply busses. Sometimes it is desirable to include either generator impedances or load impedances as series impedances in the general equations of the line. Again, it may be required to include the impedance of a shunt circuit with that of the line. The shunt may be at either end of the line or at some intermediate point, depending

[1] EVANS and SELS, *Elec. Jour.*, July, 1921: *Trans.*, A. I. E. E., pp. 33 to 35, 1924.

upon the nature of the problem. The problem rapidly becomes very complex and unwieldy as the number of elements increases.

In general, however, *equivalent constants* A_0, B_0, C_0 and D_0 can always be found, such that the supply voltage and current for the network are given in terms of the receiver voltage and current and the equivalent constants, by the equations,

$$\left.\begin{aligned} E_s &= E_rA_0 + I_rB_0 \\ I_s &= I_rD_0 + E_rC_0 \end{aligned}\right\} \tag{410}$$

The constants having the zero subscripts are constants of the equivalent simple system, replacing the more or less complex network The constant D_0 is introduced to take care of the general case, for D and A are equal only in cases in which the equivalent line is symmetrical about its center.

The generalized or equivalent line constants will be worked out for a number of cases.

a. Series Impedance Circuit.—If the circuit under consideration is a simple series impedance Z, as for example, a short transmission circuit in which the capacitance of the cables is negligible, then,

$$E_s = E_rA_0 + I_rB_0 = E_r + I_rZ$$

and

$$I_s = I_rD_0 + E_rC_0 = I_r + 0$$

whence, for this case,

$$\left.\begin{aligned} A_0 = 1,\ B_0 = Z \\ D_0 = 1,\ C_0 = 0 \end{aligned}\right\} \tag{411}$$

b. Shunt Admittance Circuit.—When the circuit consists of a shunt admittance Y, only, the voltage is the same on both sides of the admittance, but the currents on the two sides differ. Applying the general equations,

$$\begin{aligned} E_s &= E_rA_0 + I_rB_0 = E_r + 0 \\ I_s &= I_rD_0 + E_rC_0 = I_r + E_rY \end{aligned}$$

whence

$$\left.\begin{aligned} A_0 = 1,\ B_0 = 0 \\ D_0 = 1,\ C_0 = Y \end{aligned}\right\} \tag{412}$$

Using these two simple circuits as a foundation, the generalized constants for various series-parallel combinations of circuits may be evaluated.

c. Networks in Series.—Figure 63 represents two transmission lines or other networks in series. It is assumed that the constants of the separate networks are known, and that it is desired to find the constants of the equivalent simple network. The constants of the two separate components carry the subscripts 1 and 2.

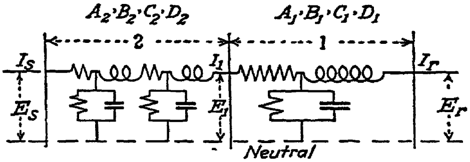

FIG. 63.—Networks in series

Beginning at the receiver end and writing the equations for voltage and current at the junction of the two networks,

$$\left.\begin{aligned} E_1 &= E_rA_1 + I_rB_1 \\ I_1 &= I_rD_1 + E_rC_1 \end{aligned}\right\} \tag{413}$$

Similarly,

$$\left.\begin{aligned} E_s &= E_1A_2 + I_1B_2 \\ I_s &= I_1D_2 + E_1C_2 \end{aligned}\right\} \tag{414}$$

By eliminating E_1 and I_1 from Eqs. (413) and (414), we find that

$$E_s = E_r(A_1A_2 + B_2C_1) + I_r(A_2B_1 + B_2D_1)$$

and

$$I_s = I_r(B_1C_2 + D_1D_2) + E_r(A_1C_2 + C_1D_2) \tag{415}$$

Therefore,

$$\left.\begin{aligned} A_0 &= A_1A_2 + B_2C_1 \\ B_0 &= A_2B_1 + B_2D_1 \\ C_0 &= A_1C_2 + C_1D_2 \\ D_0 &= B_1C_2 + D_1D_2 \end{aligned}\right\} \tag{416}$$

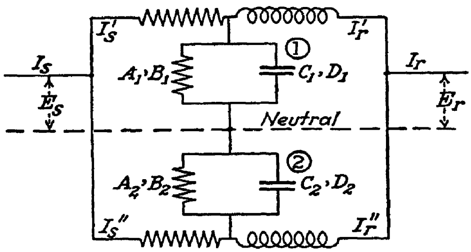

FIG. 64.—Networks in parallel.

d. Networks in Parallel.—Figure 64 represents two transmission lines or other networks in parallel. The constants of the separate networks are designated by the subscripts 1 and 2 indicated in the figure.

The drop in potential between the ends of the circuit is the same *via* either branch; hence

$$E_s = E_rA_1 + I_r'B_1 = E_rA_2 + I_r''B_2. \tag{417}$$

The currents at the supply ends of the two branches are

$$\left.\begin{aligned} I_s' &= I_r'D_1 + E_rC_1 \\ I_s'' &= I_r''D_2 + E_rC_2 \end{aligned}\right\} \tag{418}$$

However,

and
$$\left.\begin{aligned} I_r &= I'_r + I''_r \\ I_s &= I'_s + I''_s \end{aligned}\right\} \tag{419}$$

From Eq. (417),
$$I'_r = \frac{I_r B_2 - E_r(A_1 - A_2)}{B_1 + B_2}. \tag{420}$$

Eliminating I'_r from the first part of Eq. (417) and simplifying yields
$$E_s = \frac{E_r(A_1B_2 + A_2B_1)}{B_1 + B_2} + \frac{I_rB_1B_2}{B_1 + B_2}. \tag{421}$$

Adding Eqs. of (418), writing I'_r and I''_r in terms of I_r and simplifying,
$$I_s = \frac{I_r(B_1D_2 + B_2D_1)}{B_1 + B_2} + E_r\left[C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2}\right] \tag{442}$$

Accordingly, for this case,
$$\left.\begin{aligned} A_0 &= \frac{A_1B_2 + A_2B_1}{B_1 + B_2} \\ B_0 &= \frac{B_1B_2}{B_1 + B_2} \\ C_0 &= C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2} \\ D_0 &= \frac{B_1D_2 + B_2D_1}{B_1 + B_2} \end{aligned}\right\} \tag{423}$$

e. Transmission Line and Series Impedance at Supply End.—This is a special case under (*c*). The equivalent constants are given by Eqs. of (416), in which quantities having the subscripts 1 now refer to the line, while those having the subscripts 2 refer to the series impedance at the supply end. For the series impedance, Z_s, at the supply end, the generalized constants have the values given by Eq. (411). Substituting these values in Eq. (416) for the corresponding constants having the subscripts 2, and remembering that $A_1 = A$, $B_1 = B$, etc., yields
$$\left.\begin{aligned} A_0 &= A + CZ_s \\ B_0 &= B + AZ_s \\ C_0 &= C \\ D_0 &= D = A \end{aligned}\right\} \tag{424}$$

f. Transmission Line and Series Impedance at Receiver End.—This is a special case under (*c*). The equivalent constants are again given by Eq. (416), in which constants having the subscripts 1 now refer to the receiver series impedance Z_r, while those having the subscripts 2 refer to the line. Proceeding as before,

$$\left.\begin{aligned} A_0 &= A \\ B_0 &= B + AZ_r \\ C_0 &= C \\ D_0 &= A + CZ_r \end{aligned}\right\} \qquad (425)$$

g Transmission Line and Series Impedance at Both Ends—Contained in the circuit between generators and load, the usual transmission circuit includes supply-end and receiver-end trans-

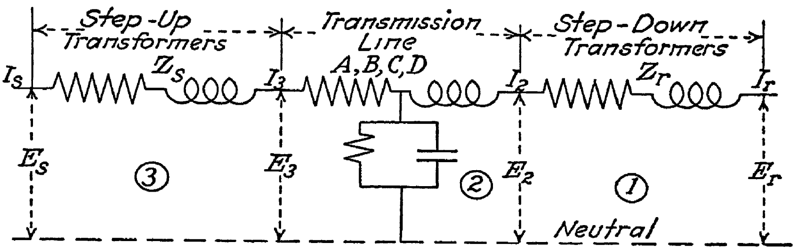

FIG 65 —Transmission line in series with transformer impedances

formers in addition to the line itself. It is therefore desirable to have equivalent generalized line constants for this case If the magnetizing currents and the losses in the transformers be neglected, the equivalent transformer circuit becomes a simple series impedance. We then have the problem of evaluating equivalent constants for a circuit consisting of the transmission line itself, plus a series impedance, Z_r, at the receiver end, plus a series impedance, Z_s, at the supply end. Z_r and Z_s may represent the receiver-end and supply-end equivalent transformer impedances, respectively. The elements of this circuit are represented in Fig. 65. This is also a special case of impedances in series and falls under (*c*).

Following the usual procedure, the equations for each section are written. Thus, for the first section,

$$\left.\begin{aligned} E_2 &= E_rA_1 + I_rB_1 \\ I_2 &= I_rD_1 + E_rC_1 \end{aligned}\right\} \qquad (426)$$

for the second section,

$$\left.\begin{aligned} E_3 &= E_2A_2 + I_2B_2 \\ I_3 &= I_2D_2 + E_2C_2 \end{aligned}\right\} \tag{427}$$

and for the third section,

$$\left.\begin{aligned} E_s &= E_3A_3 + I_3B_3 \\ I_s &= I_3D_3 + E_3C_3 \end{aligned}\right\}. \tag{428}$$

Eliminating E_2 and I_2 from Eqs (426) and (427) will yield two equations in E_3 and I_3. Using these two equations and Eq (428) and eliminating E_3 and I_3 therefrom yields two equations in E_s, I_s, E_r and I_r. Putting these resultant equations into the usual form yields the equivalent constants

$$\left.\begin{aligned} A_0 &= A_3(A_1A_2 + C_1B_2) + B_3(A_1C_2 + C_1D_2) \\ B_0 &= A_3(B_1A_2 + D_1B_2) + B_3(B_1C_2 + D_1D_2) \\ C_0 &= C_3(A_1A_2 + C_1B_2) + D_3(A_1C_2 + C_1D_2) \\ D_0 &= C_3(B_1A_2 + D_1B_2) + D_3(B_1C_2 + D_1D_2) \end{aligned}\right\} \tag{429}$$

The constants designated by the subscripts 1, 2 and 3, are already known. From (*a*) the constants with the subscripts 1 and 3 may be identified. They are

$$\begin{aligned} A_1 &= A_3 = 1 \\ B_1 &= Z_r \\ B_3 &= Z_s \\ C_1 &= C_3 = 0 \\ D_1 &= D_3 = 1. \end{aligned}$$

The constants carrying the subscript 2 are the A, B, C and D of the line itself, in which, of course, $A = D$. Substituting the above equivalents in Eqs. of (429), the resultant constants of the equivalent networks are found to be

$$\left.\begin{aligned} A_0 &= A + CZ_s \\ B_0 &= B + A(Z_r + Z_s) + CZ_rZ_s \\ C_0 &= C \\ D_0 &= A + CZ_r \end{aligned}\right\} \tag{430}$$

h. Transmission Line and Transformers at Both Ends Including Exciting Current of Transformers.—This is a case of combined series impedance and shunt admittance at each end of the line, plus the line itself. Constants for this case will not be worked

out here. These constants, if desired, may be found in the reference given at the opening of this article.

Illustrative Calculations.—It will be helpful at this point to illustrate the method of calculating the various constants of a line by the use of a numerical example. Let it be required to find the constants of a 100-mile, 60-cycle line built of No. 000 copper wire, having the following constants per mile of one conductor to neutral:

$r_1 = 0\ 326$ ohm per mile $\qquad g_1 = 0$ mho per mile
$x_1 = 0.818$ ohm per mile $\qquad b_1 = 5.24 \times 10^{-6}$ mho per mile
$l = 100$ miles.

a. Unit Line Angle $\theta = \sqrt{Y_1 Z_1}$ —

$$Z_1 = r_1 + jx_1$$
$$= 0.326 + j0.818 \text{ vector ohms}$$

or, using the polar form of notation,

$$Z_1 = z_1\underline{/\tan^{-1}\frac{x_1}{r_1}}$$
$$= \sqrt{r_1^2 + x_1^2}\,\underline{/\tan^{-1}\frac{0\ 818}{0\ 326}}$$
$$= 0\ 88057\underline{/68^\circ,\ 16\ 26'}.$$

Similarly,

$$Y_1 = 5.24 \times 10^{-6}\underline{/90^\circ,\ 0.0'}.$$

Then

$$Y_1 Z_1 = 0\ 88057 \times 5.24 \times 10^{-6}\underline{/68^\circ,\ 16.26' + 90^\circ}$$
$$= 4.6142 \times 10^{-6}\underline{/158^\circ,\ 16.26'}$$

and

$$\alpha_1 + j\beta_1 = \sqrt{Y_1 Z_1}$$
$$= \sqrt{4.6142 \times 10^{-6}}\,\underline{/\tfrac{1}{2}(158^\circ, 16.26')}$$
$$= 0.0021481\underline{/79^\circ,\ 8\ 13'}.$$

In rectangular coordinates,

$$\alpha_1 + j\beta_1 = 0.0021481\ [\cos(79^\circ, 8.13') + j \sin(79^\circ, 8.13')]$$
$$= 0.0021481(0\ 18848 + j0.98208)$$
$$= 0.00040487 + j0.0021096$$

or

$$\alpha_1 = 0.00040487$$
$$\beta_1 = 0\ 0021096$$

and

$$\alpha = \alpha_1 l = 0\ 040487 \text{ hyperbolic radian}$$
$$\beta = \beta_1 l = 0.21096 \text{ circular radian.}$$

b Surge Impedance and Surge Admittance.

$$\frac{Z_1}{Y_1} = \frac{0.88057\underline{/68°, 16\ 26'}}{5\ 24 \times 10^{-6}\underline{/90°, 0.00'}}$$
$$= 168{,}050\underline{/-90° + 68°, 16\ 26'}$$
$$= 168{,}050\underline{/-(21°, 43.74')}$$
$$Z_0 = \sqrt{\frac{Z_1}{Y_1}} = \sqrt{168{,}050}\underline{/-\tfrac{1}{2}(21°, 43.74')}$$
$$= 409.9\underline{/-(10°, 51\ 87')} \text{ vector ohms}$$

In rectangular coordinates,

$$Z_0 = 409.9\ [\cos\ (10°, 51\ 87') - j \sin\ (10°, 51\ 87')]$$
$$= 409.9(0\ 98208 - j0\ 18850)$$
$$= 402.6 - j77.27 \text{ vector ohms.}$$

The surge admittance is

$$Y_0 = \frac{1}{Z_0} = \frac{1}{409.9}\underline{/10°, 51\ 87'}$$
$$= 0.002439\underline{/10°, 51.87'}.$$

In rectangular coordinates,

$$Y_0 = 0.002439(0.98208 + j0\ 18850)$$
$$= 0\ 0023970 + j0.00045975 \text{ vector mho.}$$

c. Constants A, B and *C.*

$$A = \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta$$
$$B = Z_0 (\sinh \alpha \cos \beta + j \cosh \alpha \sin \beta)$$
$$C = Y_0 (\sinh \alpha \cos \beta + j \cosh \alpha \sin \beta)$$
$$\cosh \alpha = \cosh 0.040487 = 1.00082$$
$$\sinh \alpha = \sinh 0.040487 = 0.040498$$
$$\cos \beta = \cos\ (0.21096 \times 57.2958)°$$
$$= \cos 12°, 5.22' = 0.97783$$
$$\sin \beta = \sin 12°, 5.22' = 0.20939$$
$$A = 1.00082 \times 0.97783 + j0.040498 \times 0.20939$$
$$= 0.97863 + j0.0084799$$

that is,

$$
\begin{aligned}
a_1 &= 0.97863 \\
a_2 &= 0\ 0084799 \\
B &= (402\ 6 - j77.27)(0.040498 \times 0.97783 + j1.00082 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times 0.20939) \\
&= (402.6 - j77\ 27)(0\ 039600 + j0.20956) \\
&= 32.1357 + j81\ 3090 \\
b_1 &= 32.1357 \\
b_2 &= 81.3090 \\
C &= (0.0023970 + j0.00045975)(0.039600 + j0.20956) \\
&= -0.0000014240 + j0\ 00052052 \\
c_1 &= -0.0000014240 \\
c_2 &= 0.00052052.
\end{aligned}
$$

Physical Significance of the Constants α_1 and β_1.—The physical significance of the unit constants α_1 and β_1 is best shown by an interpretation of the exponential forms of the equations of the transmission line. These forms are obtained by substituting the expressions for the constants A, B and C of Eq (406) in Eqs (366) and (367). After rearranging, the expressions for the voltage and current l miles from the receiving end are found to be

$$E = \frac{1}{2}(E_r + I_rZ_0)\epsilon^{\alpha_1 l}\epsilon^{j\beta_1 l} + \frac{1}{2}(E_r - I_rZ_0)\epsilon^{-\alpha_1 l}\epsilon^{-j\beta_1 l} \qquad (431)$$

$$I = \frac{1}{2}(I_r + E_rY_0)\epsilon^{\alpha_1 l}\epsilon^{j\beta_1 l} + \frac{1}{2}(I_r - E_rY_0)\epsilon^{-\alpha_1 l}\epsilon^{-j\beta_1 l}. \qquad (432)$$

Since these equations are exactly alike in form, only one of them will be considered In Eq. (431) it is to be observed that the voltage E comprises the two vectors

$$E_1 = \frac{1}{2}(E_r + I_rZ_0)\epsilon^{\alpha_1 l}\epsilon^{j\beta_1 l} \qquad (433)$$

$$E_2 = \frac{1}{2}(E_r - I_rZ_0)\epsilon^{-\alpha_1 l}\epsilon^{-j\beta_1 l}. \qquad (434)$$

For a given line and for any assumed constant values of frequency, receiver voltage and load current vectors, since Z_0 is a function of the fundamental line constants and frequency only, the quantities

$$\frac{1}{2}(E_r + I_rZ_0) = M_1 \qquad (435)$$

and

$$\frac{1}{2}(E_r - I_r Z_0) = M_2 \tag{436}$$

are constant vectors. The notation of Eqs. (433) and (434) may therefore be simplified to read

$$E_1 = M_1 \epsilon^{\alpha_1 l} \epsilon^{j\beta_1 l} \tag{437}$$

and

$$E_2 = M_2 \epsilon^{-\alpha_1 l} \epsilon^{-j\beta_1 l}. \tag{438}$$

It is to be recalled that in these equations l is positive in the direction from receiving end to supply end.

At the receiver, where $l = 0$, the exponential terms in Eqs. (437) and (438) each reduce to unity, and hence

$$E_r = E = M_1 + M_2 \text{ (for } l = 0).$$

As l increases in length in the positive direction, that is, towards the supply end of the line, the component voltages E_1 and E_2 are constantly modified by the exponential operators, each in proportion to the distance l from the receiver. The operator $\epsilon^{\alpha_1 l}$ serves to stretch, or increase, the length of the component, while the operator $\epsilon^{j\beta_1 l}$ serves to rotate it in a counter-clockwise direction for positive values of l. The component vector E_2 is modified in the opposite manner. For positive values of l, the operator $\epsilon^{-\alpha_1 l}$ causes the vector to shrink, while $\epsilon^{-j\beta_1 l}$ causes it to rotate in clockwise direction, each in proportion to l.

Viewing the phenomenon from the generator end, and traveling along the line in the direction of the flow of energy, that is, towards the receiver, the generator voltage is the vector sum of the two components E_1 and E_2, which vary with the distance x from the generator and from point to point along the line. In the direction of positive values of x the component E_1 is constantly decreasing in length and falling behind in phase position (rotating clockwise), while E_2 is constantly increasing in length and gaining in phase position (rotating counter-clockwise); E_1 is the e.m.f. of a wave advancing in the direction of the energy flow, while E_2 is the e.m.f. of a reflected wave advancing in the opposite direction. Each component wave decreases in amplitude and lags increasingly in phase position as it advances.

A similar discussion holds for the current.

Attenuation and Wave Length.—From the above discussion it is apparent that the amount of shrinkage in the length of the

voltage or current vector in the direction of the traveling wave, per unit length of line, depends only upon the value of α_1. For this reason α_1 has been called the *attenuation constant* of the line.

Similarly, the amount of phase attenuation or phase shift of the vector per unit length of line depends only upon the value of β_1. The length of the line, for which the phase shift of the vector is 2π radians, is evidently one wave length λ so that

$$\lambda = \frac{2\pi}{\beta}. \tag{439}$$

β is therefore called the *wave length constant.*

On page 149 the phase shift for a certain line there considered, was found to be 0.118° per mile. The length of this line, for which the phase shift of the current and voltage vectors would be 360° (one wave length), is

$$\lambda = \frac{360}{0.118} = 3{,}050 \text{ miles (approximately).}$$

At the half wave-length point, or at 1,525 miles, the current and voltage vectors are just opposite their corresponding positions at the beginning of the line.

The complex number $a_1 + j\beta_1$, already referred to as the unit line angle, is also called the *propagation constant,* since it completely determines the manner in which the waves of current and voltage are propagated along the line.

Velocity of Propagation.—Since the number of miles of line for which the component vectors of current and voltage make one complete rotation is λ, it follows that in 1 sec. a wave will cause f complete rotations of the component vectors, and the velocity of propagation of the component waves is

$$\begin{aligned} v &= f\lambda \\ &= \frac{2\pi f}{\beta_1} \end{aligned}$$

For a line in which there are no losses, *i.e.*, for which $r = 0 = g$, $\beta_1 = 2\pi f\sqrt{L_1C_1}$—for such a line the velocity of propagation is

$$\begin{aligned} v &= \frac{2\pi f}{2\pi f\sqrt{L_1C_1}} \\ &= \frac{1}{\sqrt{L_1C_1}} \text{ miles per second} \end{aligned} \tag{440}$$

where L_1 is the inductance per mile of one line conductor, exclusive of the inductance due to the linkages within the conductor itself, and C_1 is the capacitance to neutral, per mile of line. This is the maximum attainable velocity of propagation and is equal to the velocity of light $= 3 \times 10^{10}$ cm. per second, or, approximately, 186,300 miles per second. It is also the approximate velocity of propagation of electrical impulses in ordinary well insulated high-power transmission lines, since in these lines r and g are small as compared with x and b.

The truth of the above may be verified by substituting in Eq. (440) the values of L_1 and C_1 for the case of one conductor of a parallel-sided loop suspended in air. For such a loop the inductance per mile of one conductor, neglecting internal linkages, is

$$L_1 = 0.00074113 \log_{10} \frac{D}{r}$$

while the capacitance per mile of one conductor to neutral is

$$C_1 = \frac{0.03883 \times 10^{-6}}{\log_{10} \frac{D}{r}}$$

whence, substituting and solving

$$v = 186{,}400 \text{ miles per second.}$$

Natural Frequency.—When for any given frequency the energies stored per unit length in the magnetic and dielectric fields of a line, during alternate half cycles are equal, the line is said to be in *resonance*. The frequency at which resonance occurs is called the *natural frequency* of the line.

Steinmetz has shown that for a line having uniformly distributed constants, the natural frequency is given by the equation

$$f = \frac{1}{4\sqrt{LC}} \tag{441}$$

where L is the inductance in henries of the line due to the linkages of magnetic flux with current, of the magnetic lines threading the air loop only (neglecting those within the conductors themselves), and C is the capacitance in farads of one conductor to neutral. If the constants L and C be reduced to their corresponding per mile values, the above equation becomes

$$f = \frac{1}{4l\sqrt{L_1C_1}}.$$

Remembering that the velocity of propagation is $v = \dfrac{1}{\sqrt{L_1C_1}}$, one may write

$$f = \frac{v}{4l} \tag{442}$$

and, since the wave length is $\lambda = V \div f$, it follows that, for the frequency of resonance,

$$l = \frac{\lambda}{4} \tag{443}$$

That is, a line having a length equal to one-quarter wave length for the frequency f will be in resonance to this frequency.

In order to preclude the possibility of the production of dangerously high, induced voltages, such as naturally result from resonance, it is necessary to investigate the above relations for any proposed line in order to determine whether the line is of such length that any frequency likely to be encountered during operation would be apt to produce resonance If investigation points to the possibility of the existence of dangerous frequencies, means should be found either to eliminate or suppress them.

For example, the frequency of the fifth harmonic of a 60-cycle circuit is 300 cycles per second. Resonance for this harmonic takes place in a line of length

$$l = \frac{186{,}300}{4 \times 300} = 155 \text{ miles, approximately.}$$

Therefore, a 60-cycle line, 155 miles long, would possess the possibility of giving trouble unless the line were artificially loaded, thus alternating its constants.

In transmission-line practice good engineering usually requires the use of transformer connections of a type that will make it impossible for third harmonics to exist in the line even though they may be quite prominent in the generated voltage wave. Grounded neutral systems permit triple-frequency currents to flow. Third harmonic voltages result from the magnetizing currents of transformers when Y-Y connected, or from auto transformers. For this reason Y-Y connected transformers should not be used. Triple-frequency voltages of dangerous values may be avoided by the use of suitable tertiary windings in transformers, through which the triple-frequency voltages are short-circuited, or by the use of delta-connected auxiliary transformers.

PROBLEMS

1 A 150-mile, 60-cycle, three-phase line is built of three 350,000-cir. mil, concentric-lay copper cables The line is transposed to balance phases, and the equivalent spacing is $D' = 12\ 5$ ft If the resistivity of copper at working temperature is 10.5 ohms per mil foot, find the constants, r, x, b, Z_0, Y_0, A, B and C for the line.

2. The line in Problem 1 is operated with a constant receiver voltage of 110 kv and a constant supply voltage of 120 kv. In transformers, 36,000 kva. are connected at the receiver end and 40,000 kva. at the supply end. All transformers have 0 6 per cent resistance and 10 per cent reactance. Neglecting the exciting current of the transformers, calculate the equivalent line constants A_0, B_0, C_0 and D_0 of line and transformers, referred to the high sides of the transformers.

3. If the connected generating capacity is 40,000 kva , and the generator synchronous reactance may be assumed constant and equal to 90 per cent, what are the generalized line constants including generators?

4. A certain three-phase, 60-cycle line has the following constants per mile:

$$r = 0\ 25 \text{ ohm} \qquad g = 0 \text{ mho}$$
$$x = 0\ 81 \text{ ohm} \qquad b = 5\ 1 \times 10^{-6} \text{ mho}$$

Calculate the generalized line constants per mile of line, (a) By the hyperbolic form of equations; (b) By the convergent series equations, using real numbers, and discarding all quantities involving powers of α higher than the second.

5. A line 300 miles long has the same constants per mile of line as the 150-mile line of Problem 1. The supply voltage is 100 kv. Calculate the open-circuit current and voltage for each 60-mile interval between supply-end and receiver. Plot open-circuit current and voltage as ordinates against miles from generating station as abscissas.

6. What is one wave length of a line having the constants of Problem 1? Assume a line having these constants to be 3,000 miles long. Let the receiver voltage to neutral, and the current be

$$E_r = 30{,}000 + j0 \text{ volts}$$
$$I_r = 60 - j45 \text{ amp.}$$

Calculate the voltage to neutral and the current for each 200-mile interval from receiver to generator. Plot these values against miles from receiver, (a) in rectangular coordinates, and (b) in polar coordinates.

CHAPTER IX

VOLTAGE CONTROL OF TRANSMISSION LINES

Generally speaking, alternating-current transmission systems are operated with a fixed receiver voltage. This is necessary, since approximately constant voltage is required at the distribution center at all loads. Slight variations in voltage between the distribution center and the ultimate consumer, due to load changes, are taken care of by feeder regulators.

Voltage Control by Generator Excitation.—Constancy in the value of receiver-end voltages may be maintained, within practical operating limits, by varying the excitation of the generators to suit the load and power factor conditions. This method is quite generally used in short lines. As the load in the line increases, the generator excitation is increased to the point where the additional line drop is compensated for by the additional voltage generated. Under falling loads the reverse action takes place By the use of automatic voltage regulators, properly compounded to compensate for changes in loads, and operating on the exciters of the generators, the operation of the line may be made quite automatic

While this method of voltage control is satisfactory for relatively short lines, it is not applicable to lines of great length because the required difference in voltage between no-load and maximum-load conditions becomes excessive. Considerable swings in voltage at the generator end of the line are undesirable in themselves, but even aside from this objectionable feature, a practical limit is set by saturation in the generator field.

Line Regulation.—The regulation of a transmission line is the change in voltage at the receiving end of the line between rated, non-inductive load and no load, with constant voltage impressed upon the supply end. The regulation is usually expressed as a percentage of the terminal rated voltage at the load, and, when so expressed, it is the percentage change in the receiver voltage with respect to its normal rated value.

Constant Voltage, Variable Power-factor Control.—This method of voltage control, sometimes referred to as *phase control,* is commonly used in long-line practice, as well as in some of the more important short lines. It may also be combined with the variable voltage method of control already discussed where this is found to be advantageous.

The receiver-end voltage and the supply-end voltage are both kept constant for all loads, but both receiving-end and supply-end power factors vary, as demanded by the constant voltage requirements. This is made possible by the use of synchronous reactors connected to the receiving end of the line in parallel with the load, the excitation of the reactors being automatically controlled by means of the usual type of voltage regulators.

This method of control has the advantage of maintaining constant voltages at both receiving and supply ends of the line, independent of transformer characteristics and the variable drops introduced by them and by the line, with variable loads. Furthermore, the controlling equipment, while expensive, is nevertheless simple in its operation.

Among the disadvantages of the system, the following are important: When synchronous reactors are employed at the receiving end only, as is here assumed, the voltage at any point in the line other than one of its ends, varies with the load. No constant-voltage taps are therefore possible at any intermediate point. In long lines, during light-load periods, if for some reason the synchronous reactors should fall out of step and be disconnected from the line, the voltage at the receiving end of the line would rise to dangerously high values For this reason, the line insulation of long lines should be built with considerably higher factors of safety than short-line practice would require. The high cost of the control equipment is another item of importance. Very long lines are entirely inoperative with this system of control. Such lines, where they are now operated, are usually operated as tie lines having load points connected to them at more or less frequent intervals, rather than as straight, single, transmission circuits. For very long, straight transmission lines it would be advantageous to connect synchronous reactors at one or more points along the line, as well as at the receiving end. In this way the line voltage could be held at fixed values at several points in the line, and the possibility of dangerous voltage rises would be largely avoided.

Reactive Power Required for Phase Control. Approximate Equations for Short Lines.—The discussion following is strictly applicable only to short lines for which either the so-called "impedance circuit" or the "load-end condenser circuit" applies

The equation relating to the supply and receiver voltages to neutral for these circuits is Eq. (302). If the receiver current is

$$I_r = I_1 - jI_2 \text{ vector amperes}$$

the supply voltage is

$$E_s = E_r + rI_1 + xI_2 + j(xI_1 - rI_2) \text{ vector volts.}$$

This equation applies to either the simple impedance circuit or the load-end condenser circuit, so long as I_2 is interpreted to mean the reactive component of the receiver current. Thus, for the impedance line, I_2 is the load reactive component of current, while for the load-end condenser circuit I_2 is the difference between the lagging quadrature component of the load current and the charging current of the load-end condenser. The constants r and x may be taken as the equivalent resistance and inductive reactance, respectively, of the line, including raising and lowering transformers.

Since the receiver voltage is constant and the value of the in-phase current I_1 is fixed by the receiver load, there remain only two variables in the equation, namely, the generator voltage and the reactive component I_2, of the receiver current If the latter is determined by the load alone it is beyond the control of the operator, and the receiver voltage can be held constant only by controlling the supply voltage through excitation. This gives rise to the method of voltage control by generator excitation already discussed.

On the other hand, if means are provided at the receiver end of the line by which the reactive component of the receiver current may be controlled, it is possible to keep both receiver and the supply voltages constant at predetermined values, and for all loads.• Such control is brought about by the installation of synchronous reactors at the receiver-end of the line, and the method of control employing them is called *phase control.*

The operation is about as follows: At light loads the synchronous reactor is underexcited, thus increasing the lagging reactive power of the receiver circuit and the value of I_2. The line drop is thereby increased. As the active power in the receiver circuit

rises, the lagging reactive power taken by the synchronous reactor is gradually reduced by increasing the excitation, until, at about half load for the receiver, the excitation of the reactor is increased to a point such that the reactive power of the reactor becomes leading instead of lagging, and remains leading for all greater loads. For heavy loads, therefore, much of the lagging reactive power demanded by the load is supplied by the synchronous reactor. For increasing loads the reactive component of current flowing in the line is accordingly gradually reduced from a large lagging value, and may even pass through zero and become leading if the load is increased sufficiently. The vector line drop is therefore increased under light loads and reduced under heavy loads, thus making it possible to keep the absolute values of E_s and E_r both constant for all loads.

The relation which must exist between the active load current I_1 and the reactive component I_2 of the receiver current, in order to maintain constancy in receiver-end and supply voltages, is found as follows:

$$E_s = E_r + rI_1 + xI_2 + j(xI_1 - rI_2).$$

Or, since

$$E_s = E_1 + jE_2 \text{ vector volts}$$

$$E_1 = E_r + rI_1 + xI_2 \text{ volts absolute} \tag{444}$$

and

$$E_2 = xI_1 - rI_2 \text{ volts absolute.} \tag{445}$$

Solving the simultaneous Eqs. (444) and (445) yields the relation

$$\left(I_2 + \frac{E_r x}{z^2}\right)^2 + \left(I_1 + \frac{E_r r}{z^2}\right)^2 = \frac{E_s^2}{z^2}$$

or

$$I_2 = -\frac{E_r x}{z^2} + \sqrt{\frac{E_s^2}{z^2} - \left(I_1 + \frac{E_r r}{z^2}\right)^2}. \tag{446}$$

Equation (446) may be converted to a power equation by substituting for the two components of receiver current their equivalent power expressions. These are

$$\left.\begin{aligned} I_1 &= \frac{P_r}{E_r} \\ I_2 &= \frac{Q_r}{E_r} \end{aligned}\right\} \tag{447}$$

where P_r and Q_r are the true and quadrature components, respectively, of the receiver volt-amperes. Making these substitutions and dividing through by E_r yields

$$\left(\frac{P_r}{E_r^2} + \frac{r}{z^2}\right)^2 + \left(\frac{Q_r}{E_r^2} + \frac{x}{z^2}\right)^2 = \frac{E_s^2}{E_r^2 z^2}. \tag{448}$$

Equation (448) is the equation of a family of circles in which the coordinates of the center are[1]

$$\left.\begin{aligned} l &= \frac{P_r}{E_r^2} = -\frac{r}{z^2} \\ m &= \frac{Q_r}{E_r^2} = -\frac{x}{z^2} \end{aligned}\right\} \tag{449}$$

and the radius is

$$n = \sqrt{\frac{E_s^2}{E_r^2 z^2}} = \frac{E_s}{E_r z} \tag{450}$$

where

$$z = \sqrt{r^2 + x^2}$$

the numerical value of the line impedance.

For the purpose of numerical calculations it is somewhat more convenient to multiply through by 10^6. The constants of Eq. (448) then become

$$\left.\begin{aligned} l' &= 1{,}000l \\ m' &= 1{,}000m \\ n' &= 1{,}000n \end{aligned}\right\} \tag{451}$$

and Eq. (448) takes the form,

$$\left(\frac{1{,}000P_r}{E_r^2} + l'\right)^2 + \left(\frac{1{,}000Q_r}{E_r^2} + m'\right)^2 = \frac{10^6 E_s^2}{E_r^2 z^2}. \tag{452}$$

Or, if P_r and Q_r be expressed in kilowatts and kilovolt-amperes, respectively, and E_r and E_s are in kilovolts, one may write

$$\left(\frac{P_r}{E_r^2} + l'\right)^2 + \left(\frac{Q_r}{E_r^2} + m'\right)^2 = \frac{10^6 E_s^2}{E_r^2 z^2}. \tag{453}$$

From Eqs. (448) or (453) the amount of reactive power Q_r, required at the receiver end to maintain fixed voltages at both ends of the line and bearing any assumed ratio $E_s \div E_r$ to each other, may readily be found for any assumed load, P_r.

[1] In order to make the circle diagram represent leading and lagging reactive kilovolt-amperes in the first and fourth quadrants, respectively, the centers of all circle diagrams in this book similar to this one are located at $+l$, $-m$ instead of at $+l$, $+m$, as required by Eq. (449).

Example.—As an example to illustrate the use of these equations, consider the following problem:

A line 38 miles long, built of 250,000-cir. mil, stranded, copper cable delivers a maximum load of 50,000 kw. at a power factor of 85 per cent current lagging. The receiver line voltage is 110 kv. The line constants of one conductor are

$$r = 9.03 \text{ ohms}$$
$$x = 31.0 \text{ ohms}$$
$$b = 1.98 \times 10^{-4} \text{ mho.}$$

The receiver voltage to neutral is constant and equal to $110 \div \sqrt{3} = 63.5$ kv. Assume (*a*) no charging current, and (*b*) the charging current of the entire line flowing over its full length. It is required to find the minimum synchronous reactor which will keep voltages at both receiver and supply ends of the line constant over the entire range of load for each condition, and to find the corresponding values of supply voltages.

Solution.—

$$z = \sqrt{9.03^2 + 31.0^2} = 32.3 \text{ ohms.}$$
$$z^2 = 1{,}043$$
$$\frac{1}{z^2} = 958.8 \times 10^{-6}$$
$$E_r^2 = \left(\frac{110}{\sqrt{3}}\right)^2 = 4{,}034 \text{ (for } E_r \text{ in kv.).}$$

The maximum load per phase is

$$P_r = \frac{50{,}000}{3}$$
$$= 16{,}670 \text{ kw.}$$

The maximum load reactive kilovolt-amperes is

$$Q_r = 16{,}670 \tan^{-1} (\cos 0.85)$$
$$= 10{,}330 \text{ kva. lagging.}$$

The center of the receiver power circle is at

$$l' = \frac{1{,}000 \times 9.03}{1{,}045} = -8.66$$
$$m' = \frac{1{,}000 \times 31}{1{,}043} = 29.7$$

and the radius is

$$n' = \frac{E_s}{E_r}\sqrt{958.8} = 30.96\frac{E_s}{E_r}.$$

For maximum load,

$$\frac{1{,}000P_r}{E_r^2} = \frac{16.67 \times 10^9}{4.035 \times 10^9} = 4.13.$$

For zero load,

$$\frac{1{,}000P_r}{E_r^2} = 0.$$

Substituting the above values in the equation for reactive kilovolt-amperes in the receiver circuit,

$$\frac{1{,}000Q_r}{E_r^2} = -29.7 + \sqrt{958.8\frac{E_s^2}{E_r^2} - (4.13 + 8.66)^2} \text{ for maximum load}$$
$$= -29.7 + \sqrt{958.8\frac{E_s^2}{E_r^2} - (8.66)^2} \text{ for zero load.}$$

Using values of $E_s \div E_r = 0.95$, 1.0, 1.05 and 1.1 in the above equations and solving for each of these yield the values of Q_r and Q_{sr}, as given in columns (5) and (6) of Table 9, for the case of negligible charging current.

Taking the charging current into account, it is found that the leading reactive power, introduced into the receiver circuit by it, is

$$Q_c = bE_r^2$$
$$= 1.98 \times 10^{-4} \times 4.03 \times 10^9$$

or

$$\frac{Q_c}{1{,}000} = 800 \text{ kva. leading.}$$

This has the effect of reducing by 800 kva. each of the values of Q_L in column (5), Table 9. The corrected values of Q_L for this case are shown in column (5)′ and the corresponding reactive kilovolt-amperes required in synchronous reactors in column (6)′.

TABLE 9

(1) $P_r \div 1{,}000$ = receiver kilowatts	(2) $E_s \div E_r$	(3) n'^2	(4) $Q_r \div 1{,}000$ receiver reactive kilovolt-amperes per phase required	$Q_L \div 1{,}000$ = load reactive kilovolt-amperes per phase		$Q_{sr} \div 1{,}000$ = syn reactor reactive kilovolt-amperes per phase	
				(5) No charging current	(5)′ With charging current	(6) No charging current	(6)′ With charging current
16,670	0.95	865.3	−12,910	10,330	9,530	−23,240	−22,440
0	0.95	865.3	− 6,450	0	0	− 6,450	− 6,450
16,670	1.00	958.8	− 6,050	10,330	9,530	−16,380	−15,580
0	1.00	958.8	0	0	0	0	0
16,670	1.05	1,057.1	+ 810	10,330	9,530	− 9,520	− 8,720
0	1.05	1,057.1	+ 6,450	0	0	+ 6,450	+ 6,450
16,670	1.10	1,160.1	+ 7,660	10,330	9,530	− 2,670	− 1,870
0	1.10	1,160.1	+12,910	0	0	+12,910	+12,910
16,670	1.15	1,268.0	+14,120	10,330	9,530	+ 3,790	+ 4,590
0	1.15	1,268.0	+19,360	0	0	+19,360	+19,360

Note Plus signs indicate lagging and minus signs leading reactive kilovolt-amperes

By the method explained in a succeeding article of this chapter for long lines, the best ratio of voltages is found to be $E_s \div E_r = 1.061$, approximately. For this ratio of voltages the synchronous reactor capacity required is about 7,900 kva. per phase or 23,700 kva. total for the line, assuming the reactors to be designed for equal ratings on full leading and lagging loads.

The circle diagram may readily be drawn, for the center is at $l' = -8.66$, $m' = 29.7$, and for a ratio $E_s \div E_r = 1$ the radius is

30.96. The center remains the same for all circles, and the radii for the different circles are proportional to $E_s \div E_r$. It will be noted that the sign of m' has been changed from $-$ to $+$, for the reason already given.

Reactive Power Required for Phase Control in Long Lines.—The mechanism of phase control, and the amount of reactive power, required at any given load to maintain any constant predetermined values of supply and receiver voltages, are readily understood from an examination of the equations applicable to the case. For the long line these equations are developed below:

The general vector equations of the transmission-line circuit are Eqs. (372) and (373). They are[1]

$$\left.\begin{aligned} E_s &= E_rA + I_rB \\ I_s &= I_rA + E_rC \end{aligned}\right\}$$

Using the complex notation of Eqs. (374) and (375), and writing

$$\begin{aligned} \text{Receiver current} &= I_r = {}_rI_1 - j_rI_2 \\ \text{Supply current} &= I_s = {}_sI_1 + j_sI_2 \\ \text{Supply voltage} &= E_s = E_1 + jE_2 \end{aligned}$$

the above equations become

$$E_1 + jE_2 = E_r(a_1 + ja_2) + ({}_rI_1 - j_rI_2)(b_1 + jb_2) \quad \text{supply vector volts} \tag{454}$$

$${}_sI_1 + j_sI_2 = ({}_rI_1 - j_rI_2)(a_1 + ja_2) + E_r(c_1 + jc_2) \quad \text{supply vector amp.} \tag{455}$$

Since the receiver power factor is to vary as required to keep both receiver and supply voltages constant, it is desirable to solve for ${}_rI_2$ in Eq. (454) in terms of the constants of the line, the constant voltages and the active load current ${}_rI_1$. From Eq. (454) the numerical values of the component supply voltages are

$$E_1 = E_ra_1 + {}_rI_1b_1 + {}_rI_2b_2 \text{ volts} \tag{456}$$

$$E_2 = E_ra_2 + {}_rI_1b_2 - {}_rI_2b_1 \text{ volts.} \tag{457}$$

Squaring Eqs. (456) and (457), and adding,

$$\begin{aligned} E_s^2 = E_r^2(a_1^2 + a_2^2) &+ 2E_r\,{}_rI_1(a_1b_1 + a_2b_2) \\ &+ 2E_r\,{}_rI_2(a_1b_2 - a_2b_1) + ({}_rI_1^2 + {}_rI_2^2)(b_1^2 + b_2^2). \end{aligned} \tag{458}$$

[1] If transformer impedances are to be included with the line constants, the derived constants A_0, B_0, C_0 and D_0 should be used in these equations.

Dividing Eq. (458) through by $b_1^2 + b_2^2$ and rearranging, yields

$$\left[{}_rI_1 + E_r\frac{(a_1b_1 + a_2b_2)}{b_1^2 + b_2^2}\right]^2 + \left[{}_rI_2 + \frac{E_r(a_1b_2 - a_2b_1)}{b_1^2 + b_2^2}\right]^2 = \frac{E_s^2}{b_1^2 + b_2^2}. \qquad (459)$$

This is the equation of a family of circles, there being one circle for each pair of constant values of E_s and E_r chosen (compare with Eq. (446)). For any given constant supply and receiver voltages E_s and E_r, the circle defines the reactive current ${}_rI_2$ required for any assumed value of active load current ${}_rI_1$.

When solved for ${}_rI_2$, Eq. (459) becomes

$${}_rI_2 = \frac{-E_r(a_1b_2 - a_2b_1)}{b_1^2 + b_2^2} \pm \sqrt{\frac{E_s^2}{b_1^2 + b_2^2} - \left[{}_rI_1 + \frac{E_r(a_1b_1 + a_2b_2)}{b_1^2 + b_2^2}\right]^2} \qquad (460)$$

$$= -E_rm + \sqrt{E_s^2n^2 - ({}_rI_1 + E_rl)^2} \qquad (461)$$

where

$$l = \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2} \qquad (462)$$

$$m = \frac{a_1b_2 - a_2b_1}{b_1^2 + b_2^2} \qquad (463)$$

$$n = \frac{1}{(b_1^2 + b_2^2)^{\frac{1}{2}}}. \qquad (464)$$

The coordinates of the center are

$${}_rI_1 = -E_rl \qquad (465)$$

$${}_rI_2 = -E_rm \qquad (466)$$

and the radius is

$$R = E_sn. \qquad (467)$$

Since

$$P_r = E_r\,{}_rI_1 = \text{receiver true power} \qquad (468)$$

and

$$Q_r = E_r\,{}_rI_2 = \text{receiver reactive power} \qquad (469)$$

by substituting for ${}_rI_1$ and ${}_rI_2$ in Eq. (460) their appropriate equivalents, and dividing through by E_r, the power circle is obtained. It is

$$\left[\frac{P_r}{E_r^2} + l\right]^2 + \left[\frac{Q_r}{E_r^2} + m\right]^2 = \frac{E_s^2n^2}{E_r^2}. \qquad (470)$$

The coordinates of the center are $-l$, $-m$ and the radius is $\frac{E_s n}{E_r}$.

Equation (470) is the fundamental equation of phase control, for it determines the amount of reactive power required to maintain a chosen ratio of voltages for any given receiver load P_r. Equation (459) is an exactly similar relation in terms of the component receiver currents.

Example.—An example will serve to illustrate the use of these equations. Assume a 250-mile, three-phase line, transmitting a maximum load of 45,000 kw., or 15,000 kw. per phase, at an assumed constant-load power factor of 85 per cent lagging The receiver voltage is 89 kv. to neutral, and the construction of the line is such as to yield the following derived line constants:

$$a_1 = 0.8702$$
$$a_2 = 0.0282$$
$$b_1 = 41.15$$
$$b_2 = 186\ 7$$
$$c_1 = -1\ 26 \times 10^{-5}$$
$$c_2 = 125\ 7 \times 10^{-5}.$$

It is required to find the connected synchronous reactor capacity required at the receiver to maintain the constant supply and receiver voltages in the ratios of 0 8, 0 9, 1 0, 1.1 and 1.2. The amount of capacity required is to be determined for both no-load and maximum-load conditions.

Equation (470) will be put into more convenient form for numerical work by multiplying it through by 10^6. Performing this multiplication and solving for the quantity $\frac{1{,}000Q_r}{E_r^2}$ yields

$$\frac{1{,}000Q_r}{E_r^2} = -1{,}000m + \sqrt{\frac{10^6E_s^2n^2}{E_r^2} - \left[\frac{1{,}000P_r}{E_r^2} + 1{,}000l\right]^2}$$

or, expressing P_r and Q_r in kilowatts and kilovolt-amperes, and E_s and E_r in kilovolts,

$$\frac{Q_r}{E_r^2} = -m' + \sqrt{\frac{E_s^2n'^2}{E_r^2} - \left[\frac{P_r}{E_r^2} + l'\right]^2} \qquad (471)$$

where

$$l' = 1{,}000l = 1\ 124$$
$$m' = 1{,}000m = 4\ 41$$
$$n' = 1{,}000n = 5.23$$
$$E_r^2 = (89)^2 = 7{,}921.$$

At maximum load of 15,000 kw. per phase,

$$\frac{P_r}{E_r^2} = 1.89.$$

At zero load,

$$\frac{P_r}{E_r^2} = 0.$$

Substituting the above values of the constants in Eq. (471), and solving for Q_r for each of the assumed values of the ratio $E_s \div E_r$, the values given in the following table are obtained:

TABLE 10

(1) P_r receiver kilowatts per phase	(2) $E_s - E_r$ assumed ratio	(3) $\frac{E_s^2 n'^2}{E_r^2}$	(4) Q_r receiver reactive kilovolt-amperes required per phase[1]	(5) Q_L load reactive kilovolt-amperes per phase[1]	(6) = (4) − (5) Q' synchronous reactor reactive kilovolt-amperes per phase[1]
15,000 0	0 8	17 50	−11,960	+9,290	−21,250
0 0	0 8	17 50	− 3,010	0	− 3,010
15,000 0	0 9	22 15	− 6,340	+9,290	−15,630
0 0	0 9	22 15	+ 1,270	0	+ 1,270
15,000 0	1 0	27 35	− 1,110	+9,290	−10,400
0 0	1 0	27 35	+ 5,620	0	+ 5,620
15,000 0	1 1	33 09	+ 3,880	+9,290	− 5,410
0 0	1 1	33 09	+ 9,740	0	+ 9,740
15,000 0	1 2	39 38	+ 8,630	+9,290	− 660
0 0	1 2	39 38	+13,940	0	+13,940

[1] Plus signs indicate lagging and minus signs leading kilovolt-amperes

The load reactive kilovolt-amperes, given in column (5), are:

Load reactive kilovolt-amperes

$$
\begin{aligned}
Q_L &= P_r \tan \theta_r \\
&= 15{,}000 \times 0.6196 \\
&= 9{,}290 \text{ kva lagging,}
\end{aligned}
$$

while the reactive kilovolt-amperes required of the synchronous reactor are found from (4) and (5), as indicated.

For synchronous reactors having equal ratings as generators of leading and lagging reactive kilovolt-amperes, the minimum capacity in reactors is required when the ratio of $E_s \div E_r = 1.06$, approximately. The capacity in reactors required with this ratio is approximately 7,700 kva. per phase or 23,100-kva. total.

Power-circle Diagrams.—The circle diagrams corresponding to the above analytical solution are now readily drawn. Such circles are shown in Fig. 66. They are a great aid in visualizing the line performance. They serve not only as an important check on the accuracy of the analytical work, but, if a sufficiently large scale is used, they may readily serve as a convenient substitute for the analytical solution of long line problems.

Since lagging quadrature currents are usually represented as negative quadrature quantities, while leading quadrature cur-

rents are shown as positive quantities on the quadrature or Y-axis, the corresponding reactive kilovolt-amperes will be similarly represented. Thus, lagging reactive kilovolt-amperes, while appearing in the table with a positive sign prefixed, will be shown in the circle diagram as negative quantities. The reverse will likewise hold for leading reactive kilovolt-amperes.

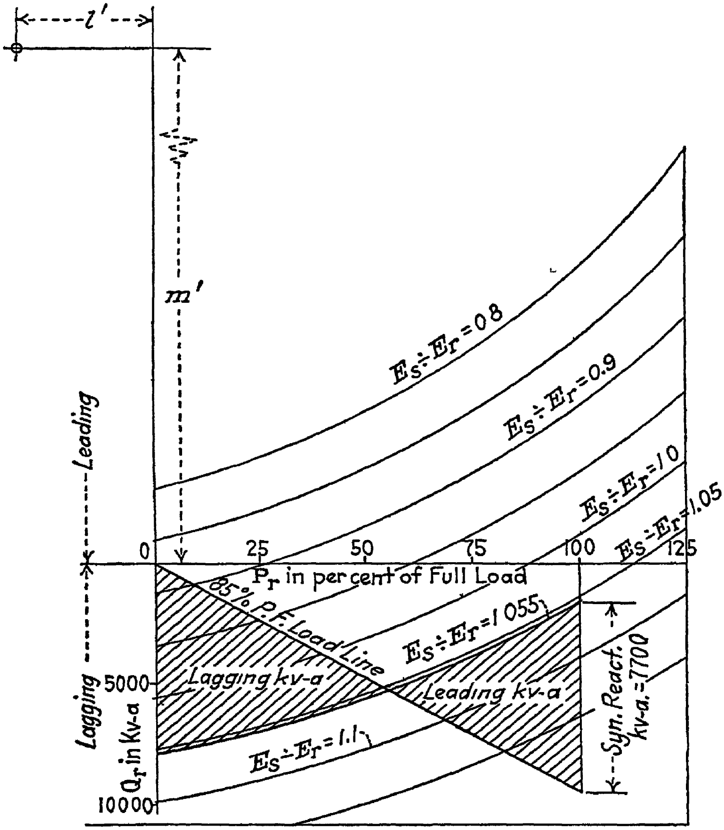

FIG. 66.—The minimum synchronous reactor capacity for transmission lines.

To do this requires that the sign of the ordinate to the center of the circle be also reversed. So, instead of the coordinates of the center appearing as $-l'$ and $-m'$, they will be taken as $-l'$ and $+m'$ for the power circles, and as $-E_r l$ and $+E_r m$ for the current circles in all diagrams where they appear.

The power circles for different ratios of supply to receiver voltages are concentric. The radius of the circle for $E_s \div E_r =$

1 is n', while for any other circle the radius is proportional to the ratio assumed; that is,

$$\text{Radius} = n' \times \text{ratio of voltages.}$$

Minimum Synchronous Reactor Capacity.—For any fixed receiver voltage, the synchronous reactor capacity, required to maintain constant voltages at both ends of the line for all loads between zero and maximum values, is a variable depending upon the ratio $E_s \div E_r$, of the voltages. That ratio which requires the minimum reactor capacity, other things being equal, is the most economical ratio to use. This ratio may be found by plotting the algebraic sums of the pairs of values of $Q' \div 1{,}000$ for no load and for maximum load and for a given ratio of $E_s \div E_r$, as a function of the ratio. Where this curve crosses the axis of $E_s \div E_r$ will determine the minimum capacity of reactors, assuming that the reactors have equal ratings as generators of leading and lagging kilovolt-amperes. If the reactors are designed for any other ratio of lagging to leading reactive kilovolt-amperes, as, for example, 60 per cent lagging and 100 per cent leading, then the required values of column (6) Table 10 should be weighted in the ratio of 6 to 10 before the sum is taken and the results are plotted.

That exactly similar diagrams may be drawn for the approximate short-line circuits is obvious.

The same results may be conveniently obtained graphically from the diagram of Fig. 66 by interpolation. For reactors having equal ratings as generators of leading and lagging reactive kilovolt-amperes, the correct ratio is found by locating the circle for which the intercept, on the Y-axis at no load, is equal to the Y-intercept between the circle and the load line at maximum load. For a 60 per cent reactor, for example, the rating, as a generator of lagging kilovolt-amperes, is 60 per cent of its rating when generating leading kilovolt-amperes, hence the above intercepts, for this case, should be in the ratio of 6 to 10 for the most economical ratio of voltages.

In the example given, the most economical ratio of voltages, for a 100 per cent reactor, is found to be 1.055, as indicated in Fig. 66.

If desired, the optimum ratio of $E_s \div E_r$ may be found analytically as illustrated in the problem of Chap. XVI.

CHAPTER X

MECHANICAL DESIGN

SPAN WITH SUPPORTS AT EQUAL ELEVATIONS

The Design of Mechanical Structure.[1]—The burden of this chapter, as indicated by the title, is to consider the transmission-line span, consisting of two supports and a cable suspended between them. To begin with, the theory will be developed for the case which assumes the points of attachment of the cable to the two supports to be at equal elevations. The case of the span, with supports at unequal elevations, will later be discussed as an extension of the theory previously developed.

Suspended Cable a Catenary.—It is well known that a perfectly flexible cord or chain of uniform structure, suspended between two supports at equal elevations, and acted on by the force of gravity only, lies in a vertical plane and assumes the form of curve called the catenary. The conductors of a transmission-line span very closely fulfil the above described conditions, for while the conductor is not entirely inelastic, yet the length of span is great as compared with the conductor diameter, so that whatever stiffness the conductor may possess has little effect in determining its position in space.

In the discussion of theory which follows, it will be assumed that the following conditions are fulfilled:

1. The suspended cable is a cylindrical solid.
2. The suspended cable is of uniform texture.
3. The suspended cable is perfectly flexible.
4. No external force except the force of gravitation acts on the cable.
5. The axis of the cable will assume the form of the catenary.

Let the curve taken by the cable be represented by the curve P_1OP_2 of Fig. 67. Since the mass is uniformly distributed along the axis of the cable, and the active gravitational forces are pro-

[1] KIRSTEN, F. K., "Transmission Line Design," *Trans.*, A. I. E. E., p. 735, 1917; University of Washington Engineering Experiment Station *Bull.* 17.

portional thereto, in a span having supports at equal elevations, there must exist a condition of symmetry of the shape of the curve P_1OP_2, with respect to a vertical plane, midway between the points P_1 and P_2 and perpendicular to the straight line P_1P_2. This plane will be selected as the reference plane YY. The point of maximum deflection of the cable from the straight line P_1P_2 must lie in this plane. A horizontal plane tangent to the curve P_1OP_2 at the point of maximum deflection is chosen as the reference plane XX. Thus O, the point of maximum deflection of the cable, is the origin of the coordinate axes.

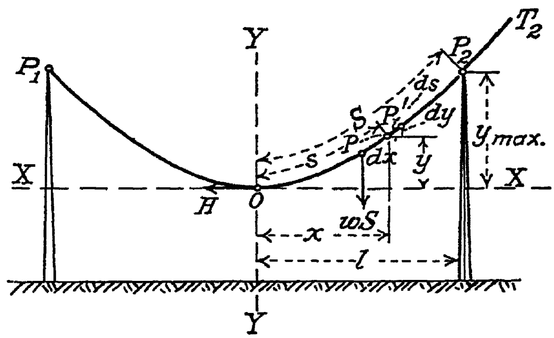

FIG 67 —Conductor suspended between supports at equal elevations

Notation.—Letters and symbols, used in the following discussion and in reference to Fig. 67, are as follows:

l = half distance between P_1 and P_2, or half tower spacing.
S = half length of suspended cable.
w = weight per unit length of suspended cable.
x and y are the rectangular coordinates of the point P.
s = the length of cable between O and P.
ds, dx and dy are increments of s, x and y, respectively, which, at the limit zero, will bear the relation $(dx)^2 + (dy)^2 = (ds)^2$.
H = tension in the cable at the point of maximum deflection.
T_2 = tension in the cable at the point of support P_2.
T = tension in the cable at P, whose coordinates are x and y.
α = angle between T and the X-axis.

Derivation of Equations.—Herein it is assumed that all stresses normal to the cross-sectional area, at any point of the cable, are concentrated on the axis of the cable and act in the direction of

the tangent to the axis at that point. Thus the slope of the curve, at any point P, is also the slope of the line of action of the tension T at that point.

The conditions for the equilibrium of the half span require that the vector sum of the forces acting be zero. The three forces in question are the horizontal tension H, the vertical load ws, and the tension T_2. If the point of support were moved to any other point such as P, the conditions for equilibrium would then require that

$$T^2 = H^2 + w^2s^2. \tag{472}$$

From the triangle of forces, it is also apparent that

$$\frac{ws}{H} = \tan \alpha \tag{473}$$

and since T is tangent to the curve at P

$$\frac{dy}{dx} = \tan \alpha \tag{474}$$

or

$$\frac{dy}{dx} = \frac{ws}{H}. \tag{475}$$

Since H is constant for a given span with given loading and at fixed temperature, and since the weight per foot of cable, w, is likewise constant under like conditions for any given material, one may write

$$\frac{H}{w} = c \tag{476}$$

where c does not vary with x and y. The constant c is evidently the length of cable whose weight is equal to the horizontal tension H.

Substituting Eq. (476) in Eq. (475),

$$\frac{dy}{dx} = \frac{s}{c}. \tag{477}$$

Since

$$(ds)^2 = (dx)^2 + (dy)^2$$

or

$$\frac{dy}{dx} = \sqrt{\left(\frac{ds}{dx}\right)^2 - 1} \tag{478}$$

by substituting Eq. (477) in Eq. (478), it follows that

$$\frac{s}{c} = \sqrt{\left(\frac{ds}{dx}\right)^2 - 1} \tag{479}$$

or

$$dx = \frac{cds}{\sqrt{s^2 + c^2}}. \tag{480}$$

By substituting in Eq (480)

$$s = c \sinh u$$

and

$$ds = c \cosh u \cdot du$$

and integrating, it is found that

$$x = cu + k \tag{481}$$

k being the constant of integration.

Since c and k do not vary with x, the constant k is seen to be zero, whence

$$\begin{aligned} x &= cu \\ &= c \sinh^{-1} \frac{s}{c} \end{aligned} \tag{482}$$

or

$$s = c \sinh \frac{x}{c} \tag{483}$$

and the length of the cable, in a half span, is the value of Eq. (483) when $x = l$

or

$$s = c \sinh \frac{l}{c} \tag{484}$$

From Eq. (475), by substituting $c = H \div w$

$$\frac{dy}{dx} = \frac{s}{c}. \tag{485}$$

Substituting the value of s from Eq. (483) in Eq (485), and transposing dx,

$$dy = c \sinh \frac{x}{c} \cdot \frac{dx}{c} \tag{486}$$

whence, by integration,

$$y = c \cosh \frac{x}{c} + k_2 \tag{487}$$

where k_2 is again the integration constant.

To determine the value of k_2, note that for $y = 0$, $x = 0$ and $\cosh \frac{x}{c} = 1$, whence $k_2 = -c$.

Finally, then,

$$y = c\left(\cosh\frac{x}{c} - 1\right). \tag{488}$$

The deflection or sag y has its maximum value when $x = l$, the half span length, so the maximum sag is

$$d = {}_{\max}.y = c\left(\cosh\frac{l}{c} - 1\right). \tag{489}$$

From Eq. (472), by substituting $c = \frac{H}{w}$ and solving for s,

$$s = \sqrt{\left(\frac{T}{w}\right)^2 - c^2}. \tag{490}$$

Substituting the value of s from Eq. (490) in Eq. (483), putting $x = l$ and solving,

$$\frac{T}{w} = c\cosh\frac{l}{c} \tag{491}$$

Since the maximum tension occurs at the point of support where $x = l$, the value of $T_2 = {}_{\max}.T$, is

$$\frac{{}_{\max}.T}{w} = c\cosh\frac{l}{c}. \tag{492}$$

Summary of Equations.—The important equations, upon which the solution of problems in span design depend, are those defining the length of cables, the sag and the tension. For convenience they are summarized below:

$$s = c\sinh\frac{x}{c} \tag{493}$$

$$y = c\left(\cosh\frac{x}{c} - 1\right) \tag{494}$$

$$\frac{T}{w} = c\cosh\frac{x}{c}. \tag{495}$$

In the above equations,

x = projection upon a horizontal plane, of the distance between the point of maximum deflection and point P.

c = length of cable whose weight is equal to the horizontal tension H.

$$\left.\begin{array}{l} T = \text{tension at point } P \\ w = \text{weight per unit length of cable} \end{array}\right\}\text{expressed in same units.}$$

x, s, y and c are all expressed in the same linear units, which is also the unit that is used with w.

Solution of Problems.—These last three equations show the concepts s, y and $T \div w$ to be hyperbolic functions of x and c, so that their magnitudes could be computed directly if both x and c were the given quantities in a problem of span design. Usually, however, c is not given since it is in reality a more or less fictitious concept, and the solution of span problems, with any two of the remaining concepts given, is accomplished by trial methods.

In order to avoid the loss of time incurred by such methods, Chart I has been devised from which, with any two of the five concepts given, the remaining three may be found at once. The chart is laid out on the basis of the decimal system to permit of easy interpolation. (Chart I is found in the pocket at the back of the book.) The quantities x and c, which form the hyperbolic argument, are the abscissas and ordinates, respectively, of the s, y and $T \div w$ curves.

Interpolation on Chart I.—It can be demonstrated that any straight line, passing through the origin (point O), is divided into intercepts of equal lengths by a set of hyperbolic curves, the indices of which vary in arithmetic progression. For instance, the curves indexed $y = 0.1$, $y = 0.2$, $y = 0.3$ . . . $y = 0.9$, $y = 1.0$ have indices which increase progressively by 0.1, and, in consequence, any straight line passing through the origin will be divided by these ten curves into ten equal intercepts. It follows, then, that, if each intercept of this straight line were again divided into ten equal lengths, the division points would be points on the curves $y = 0.01$, $y = 0.02$, $y = 0.03$ . . . $y = 0.99$, $y = 1.00$. The same reasoning holds true for interpolation between the curves $y = 1$, $y = 2$, $y = 3$, $y = 4$ . . . $y = 9$, $y = 10$ and of the last set $y = 10$, $y = 20$, $y = 30$, $y = 40$ . . . $y = 90$, $y = 100$. Similarly, this system of interpolation is correct for the s and the $T \div w$ curves.

In order to be able to accomplish quick and accurate decimal subdivision of any length of line, it is suggested to trace Fig. 68 on transparent cloth or paper for use on Chart I. Any one of the parallel lines in Fig. 68 is divided into ten, or a multiple of ten, units of equal length.

Example of Interpolation.—It is desired to interpolate for s, y and $T \div w$ a point P, the coordinates of which are $c = 550$, $x = 54$.

Through point P draw a straight line to the origin O. That this line PO is actually subdivided into equal sections by the

curves indexed in arithmetic progression is most clearly shown by the intercepts between the curves. The intercepts OA, AB, BC and CD, of the straight line formed by the curves $y = 1$, $y = 2$, $y = 3$ and $y = 4$, are exactly equal in length. The same holds true of the intercepts formed by the s and $T \div w$ curves.

By placing Fig. 68 on the chart so that the two outside radiating lines pass through points E and F and the parallel lines are at the same time parallel to line EF, point P is interpolated directly and corresponds to the magnitude $s = 54.1$.

Placing Fig. 68 so that the two outside radiating lines pass through points G and H and the parallel lines are parallel to GH, $T \div w$ is read directly and is found to be 553.

In a similar manner, y is read directly and found to be 2.65.

Chart I is independent of any fixed, conventional unit of length, weight or force, and may be used with equal precision for the

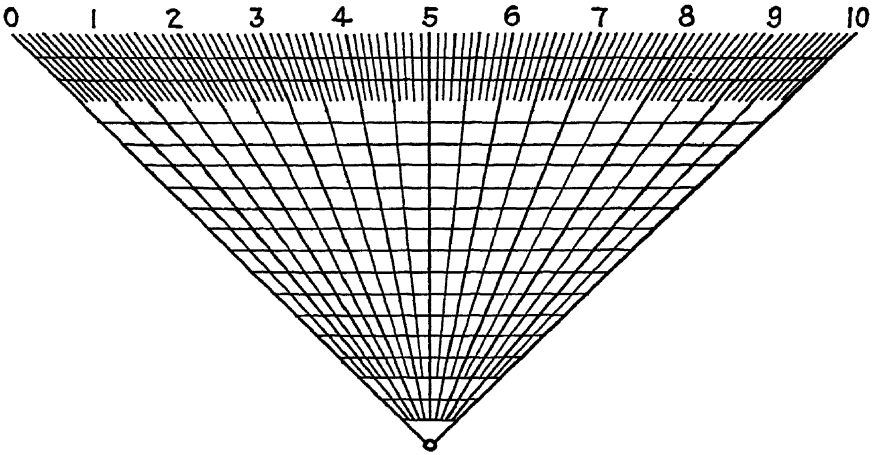

FIG. 68.—Interpolation diagram.

English foot-pound system or the decimal centimeter-gram system.

The use of the chart will now be illustrated by the solution of two representative examples.

Example 1.—A number of 40-ft. poles are to be used for supports of a line consisting of No. 00 hard-drawn, bare, copper wires. The points of support on the insulators are 30 ft. above the ground level when the poles are installed. The minimum clearance of the wire to the ground is to be 20 ft., and the tension on the wire at the point of support is not to exceed 200 lb.

What is the maximum permissible spacing of poles?

The weight per 1,000 ft. of wire is 402.8 lb.

Solution.—If the pound is chosen as the unit of force and the foot as the unit of length, $w = 402.8 \div 1{,}000 = 0.4028$, and $T \div w = 200 \div 0.4028 = 496.5$; $y = 30 - 20 = 10$.

By interpolation between curves $T \div w = 400$ and $T \div w = 500$, curve $T \div w = 496.5$ is drawn. The intersection point of this curve and the curve $y = 10$, has the abscissa $x = 98.3$, which is half the pole spacing in feet. Hence the maximum permissible pole spacing is $2 \times 98.3 = 196.6$ ft.

Example 2.—A $1\frac{1}{2}$-in steel cable is to span a river. The foundations for the anchor towers on both shores are 10 ft. above the water level and a distance of 2,000 ft apart. The maximum tension on the cable is not to exceed 70,000 lb Weight of cable is 4,700 lb. per 1,000 ft. Clearance between cable and water surface is not to be less than 50 ft.

What is the minimum height of anchor towers?

What is the length of cable between points of support?

What are the vertical and horizontal components of the tension on the points of support in the plane of the suspended cable?

Solution—Half the tower spacing is 1,000 ft. In order to bring this value within range of Chart I, the unit of length will be chosen as 20 ft. Hence $x = 1{,}000 \div 20 = 50$. For the same unit of length, $w = 4{,}700 \times 20 \div 1{,}000 = 94$. Therefore $T \div w = 70{,}000 \div 94 = 745$. This value is within range of the chart If this value had exceeded the range of the chart, a unit of length greater than 20 ft. would have had to be chosen.

Between the curves $T \div w = 700$ and $T - w = 800$, a short length of curve, $T \div w = 745$, is interpolated near the line $x = 50$ The point of intersection of this curve with the line $x = 50$ has the ordinate $c = 742$ Now, a straight line is drawn from the origin through point $x = 50$, $c = 742$, and this point interpolated on the straight line for y and for s.

Interpolation between curves $y = 1$ and $y = 2$ yields $y = 1.68$. Interpolation between $s = 50$ and $s = 60$ yields $s = 50.1$. Hence,

Maximum deflection $= y = 1.68$ units $= 33.6$ ft.

Length of cable $= 2s = 2 \times 50.1$ units $= 2{,}004$ ft.

Minimum height of anchor tower $= 33.6 + 50 - 10 = 73.6$ ft.

Vertical component of tension on point of support $= 2{,}004 \div 2 \times 4.7 = 4{,}720$ lb

Horizontal component of tension on point of support in plane of cable $= cw = 742 \times 20 \times 4.7 = 69{,}748$ lb.

Approximate Equations for Sag-tension Calculations.—For short-span calculations where the sag is only a small percentage of the span, the error, made by assuming the curve of the suspended cable to be parabolic rather than to be that of a catenary, is negligible. For spans of ordinary sags, the approximate equations applying to the parabola are probably sufficiently accurate up to spans of perhaps 80 ft. in length. For longer spans the error increases quite rapidly. It will therefore be desirable to give the approximate equations which are ordinarily used for short-span calculations, and to show wherein the approximations lie.

The three principal equations for the catenary, representing exactly the conditions prevailing in the suspended cable, have already been derived They are given by Eqs. (484), (488) and (495).

Since the total vertical stress is that due to the weight of the cable in a half span, the following relations also hold:

The vertical stress is

$$V = ws = wc \sinh \frac{l}{c} \tag{496}$$

and the horizontal stress is

$$\begin{aligned} H &= \sqrt{T_m^2 - V^2} \\ &= wc\sqrt{\cosh^2 \frac{l}{c} - \sinh^2 \frac{l}{c}} \\ &= wc. \end{aligned} \tag{497}$$

Remembering that

$$\sinh \frac{l}{c} = \frac{l}{c} + \frac{l^3}{6c^3} + \frac{l^5}{120c^5} + \frac{l^7}{5{,}040c^7} + \cdots \tag{498}$$

$$\cosh \frac{l}{c} = 1 + \frac{l^2}{2c^2} + \frac{l^4}{24c^4} + \frac{l^6}{720c^6} + \cdots \tag{499}$$

and that

$$c = H \div w \tag{500}$$

the approximate relations given below are readily found

From Eq. (483), using the first two terms of the series only, Length of cable in a half span is

$$S = l + \frac{w^2l^3}{6H^2} \text{ approximately.} \tag{501}$$

From Eq (489), using only the first two terms of the series,

$$\text{Maximum sag, } d = \frac{wl^2}{2H} \text{ approximately.} \tag{502}$$

Solving Eq. (502) for H yields

$$\text{Horizontal tension, } H = \frac{wl^2}{2d} \text{ approximately.} \tag{503}$$

Substituting the value of H from Eq. (503) in Eq. (501) yields the value of S in terms of deflection.

Length of cable in a half span is

$$S = l + \frac{2d^2}{3l}, \text{ approximately.} \tag{504}$$

Equation (504), solved for d, yields the sag. It is

$$\text{Maximum sag, } d = \sqrt{\frac{3}{2}l(S - l)}, \text{ approximately.} \tag{505}$$

The vertical stress, equal to the weight of cable in a half span, is found by multiplying Eq (504) by w, whence

$$\text{Vertical stress, } V = \frac{w}{3l}(3l^2 + 2d^2), \text{ approximately.} \tag{506}$$

The maximum stress is the vector sum of the vertical and horizontal stresses; *i e.*,

$$T_m = \sqrt{H^2 + V^2}. \tag{507}$$

Squaring Eq (503),

$$H^2 = \frac{w^2l^4}{4d^2}. \tag{508}$$

Squaring Eq. (506),

$$V^2 = \frac{w^2}{9l^2}(9l^4 + 12l^2d^2 + 4d^4). \tag{509}$$

Since d is small as compared with l, and l^2 is negligible as compared with l^4, the last two terms of Eq (509) may be neglected; whence

$$T_m^2 = \frac{w^2l^4}{4d^2} + w^2l^2$$

$$= \frac{w^2l^2}{4d^2}(l^2 + 4d^2)$$

and

$$T_m = \frac{wl}{2d}\sqrt{l^2 + 4d^2}, \text{ approximately.} \tag{510}$$

The most useful approximate equations may thus be summarized:

$$\text{Length of cable in half span, } S = l + \frac{2d^2}{3l} \tag{511}$$

$$\left.\begin{aligned} \text{Sag at any point, } y &= \frac{wx^2}{2H} \\ \text{Maximum sag, } d &= \frac{wl^2}{2H} \end{aligned}\right\} \tag{512}$$

$$= \frac{wl^2}{2T_m}, \text{ approximately.}$$

Since for small sags T and H are approximately equal,

$$\text{Maximum sag, } d = \sqrt{\frac{3}{2}l(S - l)} \tag{513}$$

$$\text{Horizontal stress, } H = \frac{wl^2}{2d}. \tag{514}$$

$$\text{Vertical stress, } V = \frac{w}{3l}(3l^2 + 2d^2) \tag{515}$$

$$\text{Maximum tension, } T_m = \frac{wl}{2d}\sqrt{l^2 + 4d^2}. \tag{516}$$

A number of semigraphical methods,[1] for calculating approximately the quantities desired in span design and based upon the above equations, have been devised. None of these will be given here, although the equations have been derived to show wherein the approximations lie.

Average Tension in Cable between Points of Support.—From Eq. (495) the tension, at any point P, is given by the relation

$$\frac{T}{w} = c \cosh \frac{x}{c}. \tag{517}$$

By the well-known rule for obtaining the average value of a variable, the average value of the ratio $T \div w$ is

$$\text{av.} \frac{T}{w} = \frac{\int_0^s c \cosh \frac{x}{c} \quad ds}{\int_0^s ds}$$

and since

$$ds = c \cosh \frac{x}{c} \cdot d\left(\frac{x}{c}\right)$$

$$\text{av.} \frac{T}{w} = \frac{\int_0^{\frac{x}{c}} c^2 \cosh^2 \frac{x}{c}\, d\, \left(\frac{x}{c}\right)}{c \sinh \frac{x}{c}}$$

$$= \frac{c}{2 \sinh \frac{x}{c}} \left[\sinh \frac{2x}{c} + \frac{x}{c}\right]$$

$$= \frac{c}{2}\left[\cosh \frac{x}{c} + \frac{\frac{x}{c}}{\sinh \frac{x}{c}}\right]. \tag{518}$$

Equation (518) will be found useful when considering the influence of changes in temperature and loading upon the tension and sag in a cable.

The Problem of Span Design.—The problem presented for solution in span design may be stated somewhat as follows: The cable in the span must be so strung that during times of severest load conditions, which occur at minimum temperature and under the assumed worst conditions of ice and wind loads, the tension in the cable will not exceed the maximum, allowable value.

[1] IMLAY, L. E., "Mechanical Characteristics of Transmission Lines. Span Formulae and General Methods of Calculation," *Elec. Jour.*, pp. 53 to 57, February, 1925.

This value is usually assumed at from 75 to 100 per cent of the elastic limit for the cable in question. (Further information on this point may be found in the handbooks.)

Since, however, the cable is presumably to be strung on the supports during fair weather, and certainly without being subjected to conditions of maximum wind and ice loads, it is necessary to predetermine the effects of both changes in temperature and changes in loading upon the tension in the cable and upon its sag An increase in temperature of the cable causes it to elongate, thus increasing its length, reducing the tension and increasing the sag. An increase in loading causes an increase in tension, an increase in length, and an increase in sag. The designer must be able to pass from one condition of load and temperature to any other condition of load and temperature, and to determine what the tension, length and sag of the cable will be under any set of circumstances. More particularly, his problem is to provide the construction foreman with charts from which he may proceed to draw up the cable to predetermined tensions or sags (depending upon which is used as the controlling factor) at known temperatures. These tensions must be such that, under the worst load conditions assumed, the added tension due to the increased load will not bring the total tension in the cable to a value in excess of the maximum tension originally allowed for the cable. Likewise, the sag under maximum load and maximum temperature under which such load may exist (32° F.), must be known so that sufficient clearance to ground may be provided for

Classes of Loading.—The maximum loadings which are assumed in the calculation of sags generally vary with the severity of climatic conditions, and, to some extent, with the judgment of the designing engineer. Standardization of assumed loadings has been proposed, and certain types of loadings have been suggested. These, however, are intended and are used as guides rather than as standards to which all designs should conform. Assumptions as to loadings for a given locality should be based upon a careful analysis of local weather conditions.

In the suggested standards for loading given below, the wind pressures, per foot of cable, are computed on the basis of the projected area of a foot length of cable. This is numerically equal to the conductor diameter in feet when there is no ice load, or to the conductor diameter plus twice the given thickness of ice, when an ice load is present. In all cases, it is assumed that

the wind load acts in a horizontal plane and is normal to the direction of the span.

Three types of loadings have been suggested by the Joint Committee on Overhead Line Construction of the National Electric Light Association, as follows:

Class A Loading.—A wind pressure of 15 lb per square foot of projected area of bare conductor plus the dead weight of the conductor.

Class B Loading.—A wind pressure of 8 lb. per square foot of projected area of conductor covered all around with ice ½ in thick, plus the dead weight of cable and ice load.

Class C Loading.—A wind pressure of 11 lb per square foot of projected area of conductor covered all around with ice ¾ in. thick, plus the dead weight of cable and ice load.

Another group of three loadings has been suggested by the National Electrical Safety Code. These are offered as appropriate loadings for the corresponding territorial belts, as indicated on the outline map of Fig. 69.

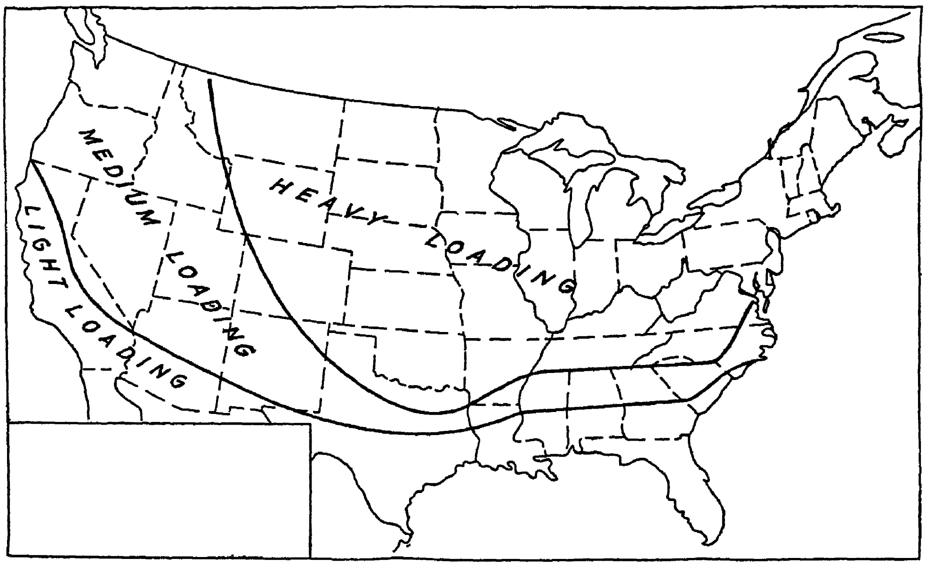

FIG. 69.—Transmission line loading map.

They are:

Heavy Loading.—A wind pressure of 8 lb. per square foot of projected area of conductor covered all around with ice $\frac{1}{2}$ in. thick, plus the dead weight of the cable and ice load. Assumed minimum temperature 0° F.

Medium Loading.—A wind pressure of 8 lbs. per square foot of projected area of conductor covered all around with ice $\frac{1}{4}$ in. thick, plus the dead weight of the cable

Light Loading.—A wind pressure of 12 lbs. per square foot of projected area of bare conductor, plus the dead weight of the conductor only The ice loading is assumed to be zero.

Wind Velocities and Pressures.—Transmission-line spans are designed to withstand considerable wind loads, in addition to the weight of the cable itself, and such sleet or snow loads as it may reasonably be expected to accumulate. It is entirely out of the question, however, to design such structures to withstand winds of the highest recorded velocities. Even if such designs were possible, it would probably be uneconomical to employ them. Aside from these exceptional storms, the worst condition prevails when the cables are covered with ice, a high wind is blowing, and the temperature is low. Lowest temperatures, highest wind velocities and snow loads do not occur simultaneously, however. Snow or sleet loads usually accumulate during calm weather. If later the temperature should fall and wind velocities should increase, the load on the structure would reach its greatest value. Very high winds would probably remove much of the snow or sleet load, and thus keep the total load from increasing further.

The wind loads which the designer assumes should take account of the above considerations. Data on wind velocities for any

TABLE 11 —WIND VELOCITIES AND PRESSURES

Velocity, miles per hour		Pressure in pounds per square foot	
Indicated velocity	Actual velocity $= V$	Projected area of cylinder $P = 0\ 0025V^2$	Flat surface $P = 0.004V^2$
10	9 6	0 23	0 4
20	17 8	0 8	1.3
30	25 7	1.7	2.7
40	33 3	2 8	4 5
50	40 8	4 2	6.7
60	48 0	5 8	9.2
70	55 2	7.6	12.2
80	62 2	9.7	15 5
90	69 2	12 0	19.2
100	76 2	14 5	23 2
110	83 2	17 3	27.7
120	90 2	20.3	32 6

given locality may be obtained from the United States Weather Bureau From such data, covering a period of years, the designer may judge what velocity it is safe to assume.

The wind velocities, as furnished by the government, are indicated velocities. The true velocities are related to these as shown in Table 11. Here are also shown the corresponding pressures per square foot of projected area on round conductors, and the pressures per square foot of flat surfaces. For flat surfaces the pressure is

$$P = 0.004V^2 \text{ lb. per square foot} \qquad (519)$$

where V is the velocity in miles per hour. For smooth cylindrical surfaces the pressure, per square foot of projected area, is one-half the above value, that is, the constant of Eq. (519) is 0.002. For stranded cable H. W. Buck found the constant to be 0.0025. Hence, for stranded cable,

$$P = 0.0025V^2 \text{ lb per square foot.} \qquad (520)$$

Influence of Changes in Temperature.—Within a certain range of temperatures, a given length of cable will change its three dimemsions practically in direct proportion to the amount of temperature change. This range more than covers the extreme range of weather conditions to which transmission spans are subjected. Accordingly, the length of a conductor may be expressed as a function of the temperature by the well known formula

$$s_t = s_0[1 + \alpha(t - t_0)] \qquad (521)$$

where

s_0 = length of cable at any given initial temperature.
s_t = length of cable at the final temperature.
t_0 = the initial temperature.
t = the final temperature.
α = the coefficient of expansion for the particular material in question.

While, as intimated above, not only the length of the cable changes with changes in temperature, but also its cross-sectional area and its weight per foot, yet the total changes in the latter two quantities, over the range of temperatures experienced, is so small, that, for practical purposes, they may be neglected.[1] In the following discussion, therefore, it will be assumed that the weight per foot and the cross-sectional area are constants, and that the length only changes with changes in temperature.

[1] For a discussion of theory, in which these variables are taken into account, see F. K. Kirsten, University of Washington Engineering Experiment Station, *Bull.* 17.

Influence of Changes in Tension.—If the tension at any point in the cable were proportional to the product of s and w, a change in temperature would not effect the tension, but, according to Eq. (491),

$$T = wc \cosh \frac{x}{c}$$

whereas, by Eq. (483)

$$s = c \sinh \frac{x}{c}.$$

Hence, a change in length, caused by either a change in temperature or a change in loading, is accompanied by a change in tension at every point in the catenary.

The condition

$$T = wc \cosh \frac{x}{c}$$

is always realized during each minute step in this change, however, the constants in the equation being x and w.

For all stresses below the elastic limit of the cable in question, the strain in the cable is proportional to the stress. This relation is expressed by Hook's Law as follows:

$$s_t = s_0\left(1 + \frac{T - T_0}{EA}\right) \tag{522}$$

where

s_0 = length of cable under initial tension
s_t = length of cable under final tension
T_0 = initial tension
T = final tension
A = cross-sectional area of cable (assumed constant)
E = modulus of elasticity for the given cable.

Since the strain is proportional to the stress, the total change in the length of the suspended cable is proportional to the change in average tension along the cable; or

$$s_t = s_0\left[1 - \frac{{}_{\text{av}}T_0}{EA} + \frac{{}_{\text{av}}T}{EA}\right] \tag{523}$$

where

${}_{\text{av}}T_0$ = the average initial tension
${}_{\text{av}}T$ = the average final tension.

Influence of Ice and Wind Loading.—In order to provide for the safety of a span, due allowance must be made for the possibility of the accumulation of additional load on the cable, due to

the formation of sleet and snow on the conductor. Furthermore, since wind pressure is likely to increase the load on the cable at any time, an allowance for wind pressure should also be made. The worst condition of loading is evidently that brought about by the simultaneous action of the increased sleet load, together with a wind load acting at right angles to the direction of the span and effective on the increased cable diameter due to the accumulated sleet.

The result of the superposition of ice and wind load upon the weight of the cable itself is to greatly increase the tension and sag in the cable, and to cause the plane of the cable to be deflected from the vertical in the direction of the wind by an angle θ, such that

$$\tan \theta = \frac{\text{wind pressure, pounds per foot of cable}}{\text{(dead weight + weight of ice load) lbs. per foot of cable}}$$

$$= \frac{p}{w + i}$$

where

$$\left.\begin{array}{l} i = \text{weight of ice load per unit length of cable} \\ p = \text{wind pressure exerted normal to the direc-} \\ \quad\ \text{tion of span on unit length of cable with} \\ \quad\ \text{ice load} \end{array}\right\} \begin{array}{c}\text{expressed} \\ \text{in the} \\ \text{same} \\ \text{unit.}\end{array}$$

The sag in the vertical plane then becomes equal to the total deflection in the inclined plane times $\cos \theta$, while the deflection horizontally is the total deflection in the inclined plane times $\sin \theta$. Unless otherwise stated, where sags are mentioned, the total deflections in the plane of the resultant force are given.

The resultant force, per unit length of cable and under conditions of combined wind and ice load, acts downwards in the plane of the deflected cable, and has the value

$$w_1 = [(w + i)^2 + p^2]^{\frac{1}{2}}. \qquad (525)$$

All of the factors which affect the suspended cable under varying conditions of loading and temperature have now been considered, together with the theory required to translate these influences into changes of length, tension and sag in the curve of the suspended cable. It remains only to summarize the results in concise mathematical form for convenient use in the solution of such problems as ordinarily arise in span design.

Let it be therefore required to find all possible catenaries that may be described by a cable under any possible condition of

loading and temperature. The characteristics of the cable are made available from the following data, assumed to be furnished:

w = weight per unit length of cable, A = cross-sectional area of the cable — assumed constant at all temperatures.
E = modulus of elasticity.
α = coefficient of linear expansion.
T_m = maximum allowable tension for the cable.

From the records of the local weather bureau, maximum and minimum temperatures and wind velocity to be assumed in making calculations, may be estimated. These are

t_1 = minimum assumed temperature at which sleet and wind loads exist simultaneously.
t_2 = maximum assumed temperature of conductor
v = maximum assumed wind pressure, per unit of area normal to the direction of wind.

Catenary Covering Conditions at Minimum Temperature. Cable under Ice Load and Wind Pressure.—Subscripts 1 refer to the above described conditions at minimum temperature t_1 Under the conditions described, the cable is assumed to be stressed to its maximum allowable tension at the points of support. By Eq. (491), the maximum tension is

$$_{\max} T_1 = w_1 c_1 \cosh \frac{x}{c_1}. \tag{526}$$

As already pointed out, the amount of the allowable maximum tension is known from available data, and, since w_1 may be calculated for any assumed ice and wind loads, Eq (526), may be solved for any assumed values of $\frac{x}{c_1}$.

If the ice covering on the cable be assumed to be a units in uniform thickness, then the volume of ice per unit length of cable is

$$\text{Volume of ice per unit length} = \pi a(d_s + a) \tag{527}$$

where d_s is the diameter of stranded cable used.

The weight of ice per unit length of cable is

$$i = u\pi a(d_s + a) \tag{528}$$

where u is the weight per unit volume of ice.

The force, exerted by the wind on unit length of cable with ice envelope, is

$$p = v(d_s + 2a). \tag{529}$$

The result, given in Eq. (529), is based on the assumption that the wind pressure on a unit length of the conductor is equal to that which would be experienced by its projected area on a plane normal to the direction of the wind.

Assuming that the wind pressure acts in a horizontal plane and in a direction normal to the direction of the span, the total resultant weight, per unit length of conductor in the inclined plane and by Eq. (525), is found to be

$$w_1 = \{[w + u\pi a(d_s + a)]^2 + v^2(d_s + 2a)^2\}^{\frac{1}{2}}. \tag{530}$$

By substituting the value of w_1 as calculated from Eq (530), in Eq. (526), the values of ${}_{\max}T_1 \div w_1$, corresponding to any number of assumed values of $\frac{x}{c_1}$, may be calculated.

By Eq. (526),

$$c_1 = \frac{{}_{\max}T_1}{w_1} \div \cosh\frac{x}{c_1},$$

and, since for any assumed values of $\frac{x}{c_1}$ the quantity $\cosh\frac{x}{c_1}$ may be found from tables, the value of c_1 itself is readily computed. The corresponding values of length of cable are obtained from Eq. (483), by which

$$s_1 = c_1 \sinh\frac{x}{c_1} \tag{531}$$

while, from Eq. (488), the sag is

$$y_1 = c_1\left(\cosh\frac{x}{c_1} - 1\right) \tag{532}$$

which may readily be transformed to

$$y_1 = \frac{{}_{\max}T_1}{w_1} - c_1. \tag{533}$$

Equations (526), (531), and (532) or (533) completely determine the catenary under the conditions of minimum temperature and maximum loading and serve as the starting point in span design. It will presently be shown how the equations of the catenaries, for any other possible condition of loading and for

any other temperature, are derived, using these equations, together with Eqs. (521) and (522) as the basic relations.

Catenary Covering Any Condition of Loading and Any Temperature.—If the temperature of the conductor could increase without at the same time altering the tension in it, the elongation of the conductor would be proportional to the temperature change. Thus, assuming that the temperature increases from the minimum value t_1 to any other value t_n, but that the tension does not change, the new length of conductor becomes

$$s_1' = s_1[1 + \alpha(t_n - t_1)]. \tag{534}$$

Owing to the change in temperature, however, a change in tension takes place throughout the conductor, beginning with the initial value ${}_{\text{av}}T_1$ at minimum temperature, and ending with some other value. Furthermore, if the change in temperature is great enough, that is, if the final temperature lies above the freezing temperature, the ice load will drop off, and a further change in tension will take place due to the change in loading. The load may also change due to the dying down of the wind even at temperatures below freezing. But, in any case, if the final loading per unit length of conductor is w_n and the average tension corresponding to the new conditions of loading and temperature is ${}_{\text{av}}T_n$, the initial and final conditions must separately satisfy Eq (518). Therefore, initially

$${}_{\text{av}}T_1 = \frac{c_1 w_1}{2}\left(\cosh\frac{x}{c_1} + \frac{\frac{x}{c_1}}{\sinh\frac{x}{c_1}}\right) \tag{535}$$

and, finally,

$${}_{\text{av}}T_n = \frac{c_n w_n}{2}\left(\cosh\frac{x}{c_n} + \frac{\frac{x}{c_n}}{\sinh\frac{x}{c_n}}\right) \tag{536}$$

Due to the change in tension which has been induced either by change in loading or change in temperature, or both, a further change in length of the cable takes place, as determined by Eq. (523). That is, the length s_n, under final conditions of temperature and loading, is

$$s_n = s_1\left(1 - \frac{{}_{\text{av}}T_1}{EA} + \frac{{}_{\text{av}}T_2}{EA}\right) \tag{537}$$

or, since, from Eq. (483)

$$s_n = c_n \sinh \frac{x}{c_n}, \tag{538}$$

$$c_n \sinh \frac{x}{c_n} = s_1'\left(1 - \frac{{}_{\text{av}}T_1}{EA} + \frac{{}_{\text{av}}T_2}{EA}\right). \tag{539}$$

The substitution of Eqs. (534), (535), and (536) in Eq. (539) yields the equation from which the catenary under the final conditions of temperature and loading may be calculated.
Making these substitutions yields

$$c_n \sinh \frac{x}{c_n} = s_1[1 + \alpha(t_n - t_1)]\left\{1 - \frac{c_1 w_1}{2EA}\left(\cosh \frac{x}{c_1} + \frac{\frac{x}{c_1}}{\sinh \frac{x}{c_1}}\right) + \frac{c_n w_n}{2EA}\left(\cosh \frac{x}{c_n} + \frac{\frac{x}{c_n}}{\sinh \frac{x}{c_n}}\right)\right\} \tag{540}$$

Dividing Eq. (540) through by x yields

$$\frac{\sinh \frac{x}{c_n}}{\frac{x}{c_n}} = \frac{s_1}{x}\left[1 + \alpha(t_n - t_1)\right] - \frac{w_1 s_1}{2EA}\left[1 + \alpha(t_n - t_1)\right] \times \left[\frac{\cosh \frac{x}{c_1}}{\frac{x}{c_1}} + \frac{1}{\sinh \frac{x}{c_1}}\right] + \frac{w_n s_1}{2EA}\left[1 + \alpha(t_n - t_1)\right]\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right] \tag{541}$$

$$= F_n - G_n + H_n\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right] \tag{542}$$

where, to simplify the notation, the following abbreviations are made:

$$D_n = 1 + \alpha(t_n - t_1)' \tag{543}$$

$$N = \frac{s_1}{2EA} \tag{544}$$

$$P_1 = \left[\frac{\cosh \frac{x}{c_1}}{\frac{x}{c_1}} + \frac{1}{\sinh \frac{x}{c_1}}\right]. \tag{545}$$

Thus, in Eq. (542),

$$F_n = \frac{s_1 D_n}{x} \tag{546}$$

$$G_n = w_1 P_1 N D_n \tag{547}$$

$$H_n = w_n N D_n. \tag{548}$$

Recapitulation.—In the design of a span, the engineer must guarantee (*a*) that the tension in the suspended cable will not exceed a predetermined safe value when subjected to the maximum loading and minimum temperature conditions; and (*b*) that the clearance of conductor to ground will never be less than a given predetermined amount.

The first requirement is met by the use of Eq (526), in which the maximum tension ${}_{\max}T_1$ is the maximum safe stress on the cable, and w_1 is the total resultant unit loading on the cable, including weight of cable, weight of ice load and wind load normal to these. The second requirement must be satisfied from an investigation of the vertical sag of the cable under the extreme conditions A little reflection will disclose what these extreme conditions are.

Minimum and Maximum Sags.—It is evident that the minimum vertical sag occurs either when the temperature is minimum and the cable is free from wind and ice loads, or when, at minimum temperature and maximum loading, wind pressure deflects the cable from a vertical plane. If, at minimum temperature, the cable receives its maximum assumed ice load and, at the same time, is subjected to its assumed wind load, the tension in the cable is increased and the cable is stretched due to the increased tension. Since the distance between supports remains fixed, the stretch in the cable gives rise to an increased sag. If the loading now remains unchanged but the temperature increases, the cable is further elongated. But as the temperature rises and the cable elongates, the tension is diminished and the cable tends to shorten due to the lessened tension. For all materials now used as transmission-line conductors, however, the increase in length, due to any increment of temperature increase, is greater than the corresponding decrease in length which accompanies the associated decrement of tension. These two conflicting forces thus actually result in an elongation of the cable. If the temperature rise should continue indefinitely and the total load on the cable did not change, a condition of maximum sag would be

found at the highest temperature at which the ice load could exist, that is, at 32° F. Thus, the sag of the cable with maximum load and freezing temperature must be found, since it may prove to be a critical condition This sag is found from the equation

$$y_m = c_n\left(\cosh \frac{x}{c_n} - 1\right)$$

after $\frac{x}{c_n}$ has been evaluated from Eq (489) for $t_n = 32°$ F. and $w_n = w_1$.

The sag found in this way will be measured in the plane of the resultant force, unless, as is sometimes done, it be assumed that, at freezing temperature, the wind has died down to a negligible value, but that the snow and ice load may have increased to an amount such that the total load per unit length of cable remained unchanged. The sag would then be in a vertical plane. In any case, if y_m is the sag in the plane of the resultant force, the vertical component of sag is given by the equation

$$_{\max} y_v = {}_{\max} y \cos \theta$$

where

$$_{\max} y_v = \text{the vertical component of sag}$$

and

θ = the angle which the plane of the resultant deflection makes with a vertical plane.

As soon as the temperature rises above 32° F., the ice load and a large part of the wind load vanish, resulting at once in decreased deflection. If the temperature should now increase to its maximum, the deflection will again increase to a new maximum, which may be either smaller or larger than the deflection at freezing point and maximum load, depending upon the relative magnitudes of ice and wind loading existing at the freezing point, and upon the maximum temperature. Solving Eq. (541) for the value of $\frac{x}{c_n}$ corresponding to $t_n = {}_{\max}t$ and $w_n = w$ (assuming no wind load to exist at the maximum temperature), the value of the sag at maximum temperature may be found from the equation for sag, namely,

$$y_n = c_n\left(\cosh \frac{x}{c_n} - 1\right)$$

Critical Catenary.—A comparison of the deflection at maximum temperature, with the deflection at freezing temperature and

maximum load, will determine which sag is the greatest. The catenary having the greatest deflection is called the *critical catenary.* In warm climates the critical catenary will tend to exist at the maximum temperature, whereas in cold climates it is likely to be found at the freezing point. The diameter of the cable used is also a factor in the determination of the critical catenary, as are also the wind and ice loadings assumed. The smaller the cable diameter the greater is the ratio of the combined unit length wind and ice load to the weight of a unit length of the cable itself. Accordingly, in a given locality the critical catenary may occur at the maximum temperature for a large cable, and at the freezing point for a smaller cable. It is therefore desirable to calculate the deflections for both conditions in order to be certain which one it is that controls.

Method of Solving Eq. (541) for $\frac{x}{c_n}$.—While Eq. (541) appears rather cumbersome and difficult to manipulate, yet when viewed in the light of its more simplified form, as given in Eq (542), and not losing sight of the fact that the problem to be solved is a very complex one, the relative simplicity of the solution will become apparent.

The solution of this equation is easily and rapidly obtained by the use of the curves of Chart II, which is found in the pocket on the inside of the back cover. The curve indexed

$$\left[\sinh\frac{x}{c} \div \frac{x}{c}\right] - 1$$

gives values of the magnitude

$$\left[\sinh\frac{x}{c} \div \frac{x}{c}\right] - 1$$

for arguments of $x \div c$ ranging from 0 to 0.25. The curve indexed

$$0.0001 \times \left[\frac{\cosh\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh\frac{x}{c}}\right]$$

represents magnitudes of

$$H \times \left[\frac{\cosh\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh\frac{x}{c}}\right]$$

for the same range of the arguments, and $H = 0.0001$. Corresponding curves, for values of H ranging from 0 to 0.0002, are indexed 0.00001, 0.00002, . . . 0.00019, 0.00020. A straight line, normal to the axis of $x \div c$ is divided into equal lengths by these curves indexed in arithmetic progression. Consequently, direct interpolation may be effected along such lines perpendicular to the axis of $x \div c$.

For all positive arguments, the ratio of the hyperbolic sine to the argument is always greater than unity. By subtracting unity from this ratio, the two sets of curves of Chart II are located nearer the axis of $x \div c$. The equation is again balanced by subtracting unity from the constants. Hence, for convenience, Eq. (542) is changed in form to

$$\frac{\sinh \frac{x}{c_n}}{\frac{x}{c_n}} - 1 = F_n - G_n + H_n\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right] - 1.$$

Supposing $F_n = 1.002821$, $G_n = 0.000525$ and $H_n = 0.0000315$, then

$$F_n - G_n - 1 = 0.002296.$$

The magnitude

$$H_n\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right] = 0.0000315\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right]$$

is given by a curve interpolated between curves indexed 0.00003 and 0.00004 for any argument $x \div c_n$ between the limits 0 and 0.25. The value 0.002296 added to an ordinate of this curve must, according to the above equation, be an ordinate of curve

$$\left[\sinh \frac{x}{c_n} \div \frac{x}{c_n}\right] - 1.$$

Hence, the correct argument $x \div c_n$, which gives the same length of ordinate for the curves

$$\left[\sinh \frac{x}{c_n} \div \frac{x}{c_n}\right] - 1$$

and

$$0.002296 + 0.0000315\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right]$$

can be found by marking the length 0.002296 on the straight edge of a sheet of paper and by moving the point marked on the sheet along the curve 0 0000315, keeping the straight edge perpendicular to the $x \div c_n$ axis, until the other point touches the curve

$$\left[\sinh \frac{x}{c_n} \div \frac{x}{c}\right] - 1.$$

The straight edge in this position indicates, on the $x \div c$ axis, the value 0.1294, which, if substituted for $x \div c_n$ in the above equation, will satisfy this equation

The curves of Chart II have been drawn with great care from hyperbolic functions computed accurately to the required number of decimal places. With a little practice in interpolation, the argument of Eq. (542) can be found accurately to the fourth decimal place. Values of

$$0.0001\left[\frac{\cosh \frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh \frac{x}{c}}\right]$$

with a given argument, can be read directly to the sixth decimal place from the curve bearing this index. For most practical design problems, the degree of accuracy possible with the use of the chart is more than ample.

If greater accuracy is required than can be obtained by direct reading of Chart II, this chart may be magnified to any desired degree by the use of the tables (Appendix C) of hyperbolic functions of $\frac{x}{c}$, calculated accurately to the tenth decimal place

The functions

$$\frac{\sinh \frac{x}{c}}{\frac{x}{c}} - 1$$

and

$$\frac{\cosh \frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh \frac{x}{c}}$$

are tabulated so that small portions of a chart similar to Chart II may be quickly drawn with ordinates of any desired magnitude, and used for the solution of Eq (542), with much greater precision than can be obtained by the use of Chart II only. Chart II should be used to determine the approximate value of $\frac{x}{c}$ to the third decimal place From the tables two or three values, greater and smaller than $x \div c$, should be plotted for the functions

$$\frac{\sinh\frac{x}{c}}{\frac{x}{c}} - 1 \text{ and } H\left[\frac{\cosh\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh\frac{x}{c}}\right] \pm (F - G - 1)$$

upon coordinate paper with ordinates of a magnitude which will yield the desired precision. Since greatly magnified, short portions of these curves appear to be almost straight lines, the intersection point of the curves may be very accurately located

Temperature-tension Stringing Charts.—The construction engineer should have a means of quickly finding the required stringing tension for any temperature and for the loading corresponding to the weight of conductor alone. The stringing tension, as already explained, should be such that under minimum temperature and maximum loading the cable will not be stressed beyond its maximum safe allowable value. Information as to the proper stringing tension should be available for all lengths of spans which may be required in the construction, and for all temperatures likely to be encountered while the construction work is in progress.

In order to supply this necessary information, the curves which the conductor takes at each of a number of different temperatures should be computed for each of a number of values of the argument $x \div c$. The range in the argument should be great enough to embrace all of the span lengths required. Data for curves of stringing tensions T vs. $x \div c$, and corresponding sags y vs. $x \div c$, are computed for a limited number of different temperatures. The range covered by the extremes in temperature should, of course, include all temperatures likely to be encountered during construction. Curves are then drawn for the constant temperatures chosen. Tensions and sags for other temperatures may be found from the curves by interpolation. The charts, consisting of tension and sag curves vs. span lengths,

similar to Fig. 70, are referred to as the *temperature-tension stringing charts.*

The use of the theory developed in this chapter will best be understood by applying it to the solution of a definite, illustrative problem. Such a problem is stated in the following example.

Example —A 500,000-cir mil copper cable is to be suspended from supports at equal elevations in spans of lengths lying between 200 and 1,200 ft. (Loading and other constants are as given in the tabulation of specifications and constants) It is required to compute the data for and to furnish a temperature-tension stringing chart to include the following curves (*a*) cable under maximum load at 32° F ; (*b*) cable without ice or wind loads at 10°, 30°, 60° and 100° F.

Solution —In solving the problem it will be convenient first to tabulate, in systematic order, all of the required data pertaining to it, as illustrated in Table 12, entitled Specifications and Constants, for the 500,000-cir mil copper cable

TABLE 12 —SPECIFICATIONS AND CONSTANTS

500,000-Cir Mil, Stranded, Copper Cable

1	Minimum temperature	t_n =	−10° F
2	Freezing temperature	t_n =	+32° F
3	Maximum temperature	t_n =	+100° F
4	Ice loading (thickness all around cable)	a	0 5 in
5	Wind pressure normal to span (pounds per square foot)	v	10
6	Weight of ice (pounds per cubic inch)	u	0 0332
7	Modulus of elasticity	E	16,000,000
8	Coefficient of linear expansion	α	9 22 × 10^{-6}
9	Elastic limit, stranded, copper cable (pounds per square inch)	T_e	28,350
10	Weight of copper (pounds per cubic inch)	W	0 327
11	Cable area in circular mils	cir mil	500,000
12	Number of strands		37
13	Number of layers around central wire		3
14	Circular mils per strand = (11) − (12)		13,513.513
15	Diameter of strand in inches = $\sqrt{(14)}$ − 1,000		0 11625
16	Outside diameter of cable in inches = [2 × (13) + 1] × (15)	d_s	0 81375
17	Diameter of equivalent solid rod = $\sqrt{(11)}$	d	0 70711
18	Area of equivalent solid rod = [(11) × 10^{-6} × π] − 4	A	0 3927
19	Area of strand in square inches = [(14) × 10^{-6} × π] − 4		0 0106135
20	Elastic limit per strand in pounds = (19) × (9)		300 8927
21	Elastic limit of cable in pounds = (12) × (20)		11,133 02
22	Maximum allowable tension assumed = 0 75 × (21)	T_m	8,349 76
23	Total modulus of elasticity for cable = (7) × (18)	EA	6,283,200
24	Weight per foot of cable alone = (18) × 12 × (10)	w	1 541
25	Cubic inches ice load per foot of cable = $12\pi[(16) + 2a)^2 - (16)^2] - 4$		24 764
26	Ice load per foot of cable in pounds = (25)u	i	0 822
27	Wind load per foot of cable in pounds = (5) × [(16) + 2a] − 12	p	1 511
28	Total resultant load in pounds per foot of cable = $[(w + i)^2 + p_2]^{\frac{1}{2}}$	w_1	2 804

Next assume a number of arguments, as in column (1) of Table 13, of sufficient range to cover the spans desired. From the tables of Appendix C, column (2) readily follows Since

$$T_m = w_1 c_1 \cosh \frac{x}{c_1},$$

in which equation all quantities except c_1 are now known, c_1 of column (3) may be computed. The corresponding half-span lengths in column (4) are

$$x = \frac{x}{c_1} \times c_1$$

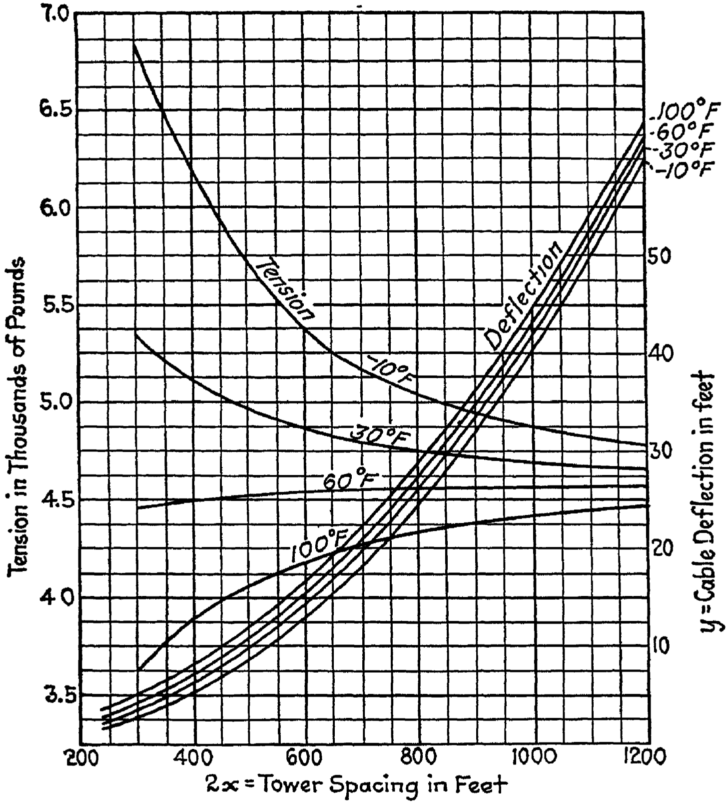

FIG. 70 —Temperature-tension stringing chart for 500,000-circular mil copper cable (See columns 18 and 19 of Tables 15 to 18.)

and the length of the loaded cable, for each assumed argument, is

$$s_1 = c_1 \sinh \frac{x}{c_1}$$

as in column (5). If desired, the sag under the condition of maximum loading and minimum temperature may also be found from the equation (532)

$$y_1 = c_1\left(\cosh \frac{x}{c_1} - 1\right)$$

TABLE 13.—CATENARY AT MINIMUM TEMPERATURE

Cable under Maximum Loading Conditions

Data and working equations

$T_m = 8{,}350$ lb
$EA = 6{,}283{,}200$
$w = 1\ 5410$ lb
$w_1 = 2\ 8034$ lb
$t_1 = -10°$ F

$$\frac{T}{w} = c \cosh \frac{x}{c},$$

$$c = \frac{T}{w \cosh \frac{x}{c}}$$

$$S = c \sinh \frac{x}{c}$$

Sags are not given since they are not critical

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Assumed argument	From tables	$\frac{T_m}{w_1} \times (2)$	$(1) \times (3)$ $x =$	$(3) \times \sinh (1)$ $s_1 =$	$(5) - (4)$	From tables $P_1 =$	$[(5) - 2EA] \times 10^5$	$w_1 \times (7)$ $w_1P_1 =$
$\frac{x}{c_1}$	$\frac{1}{\cosh \frac{x}{c_1}}$	c_1	$\frac{x}{c_1} \times c_1$	$c_1 \sinh \frac{x}{c_1}$	$\frac{s_1}{x}$	$\frac{\cosh \frac{x}{c_1}}{\frac{x}{c_1}} + \frac{1}{\sinh \frac{x}{c_1}}$	$N \times 10^5$	$2\ 8034 \times (7)$
0 04	0 999200	2,976 02	119 041	119 073	1 000265	50 013338	0 91755	140 2074
0 06	0 998202	2,973 04	178 380	178 489	1 000613	33 353316	1 42036	93 5028
0 08	0 996808	2,968 89	237 511	237 765	1 001069	25 026698	1 89207	70 1598
0 10	0 995021	2,963 57	296 357	296 851	1 001667	20 033394	2 36226	56 1616
0 12	0 992813	2,957 08	354 850	355 702	1 002401	16 706772	2 83058	46 8358
0 14	0 990279	2,919 45	112 923	111 273	1 003269	11 332519	3 29667	40 1799
0 16	0 987335	2,910 68	470 509	172 519	1 004272	12 553581	3 76017	35 1927
0 18	0 984016	2,930 79	527 542	530 396	1 005110	11 171467	1 22074	31 3181
0 20	0 980328	2,919 81	583 962	587 863	1 006680	10 067155	1 67805	28 2223

TABLE 14.—CATENARY AT FREEZING POINT

Cable under Maximum Loading Conditions

$w_1 = 2.8034$ lb

$t_n = +32°$ F

$D_n = 1 + 9.22 \times 10^{-6}(32 + 10)$

$= 1.0003872$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
$(6) \times D_n$ F	$(8) \times D_n$ $\frac{10^6 w_1 D_n}{2EA} =$ $10^6 N D_n$	$(9) \times (11)$ G	$\frac{w_1 \times (11)}{10^6}$ H	$(10) - (12) - 1$ $F - G - 1$	From chart $\frac{x}{c_n}$	$(4) - (15)$ c_n	cosh (15) $\cosh \frac{x}{c_n}$	$w_1(16)(17)$ $T =$ $w_1 c_n \cosh \frac{x}{c_n}$	$[(18) - w_1] - (16)$ sag[1] $=$ $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1 000652	0 947917	0.0013290	0 00002657	−0 000677	0 0491	2,409 7	1.00122	6,763 6	2 9
1 001000	1 42091	0 0013286	0 00003983	−0 000329	0 0698	2,555 6	1 00244	7,181 9	6 3
1 001457	1.89280	0 0013280	0 00005306	+0 000129	0 0892	2,662 7	1 00398	7,491 3	10 6
1 002055	2 36317	0 0013272	0 00006625	0 000728	0 1083	2,736 4	1 00587	7,716 3	16 1
1 002789	2 83168	0 0013262	0 00007938	0 001463	0 1275	2,783 1	1 00814	7,865 7	22 7
1 003657	3.29795	0 0013251	0 00009246	0 002332	0 1465	2,818 6	1 01075	7,986 6	30 3
1 004661	3 76163	0 0013238	0 00010545	0 003337	0 1660	2,834 4	1 01381	8,055 7	39 1
1 005799	4 22237	0.0013224	0 00011837	0 004477	0 1858	2,839 3	1 01731	8,097 5	49 2
1.007070	4 67986	0 0013208	0 00013120	0 005749	0 2051	2,847 2	1 02111	8,150 3	60 1

[1] The sags given in this and succeeding tables are in the plane of the resultant force. The sag in the vertical plane is obtained by multiplying these values by $\cos \theta = 0.8427$, where $\tan \theta = \frac{\text{wind load per foot of cable}}{(\text{ice load} + \text{weight of conductor}) \text{ per foot of cable}} = \frac{1.511}{2.363} = 0.6395$

TABLE 15 —CATENARY AT −10°F.
Cable without Ice or Wind Load

$w = 1\ 5410$
$t_n = -10°\ F$
$D_n = 1 + 9\ 22 \times 10^{-6} \times 0$
$= 1$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
$(6) \times D_n$ F	$(8) \times D_n$ $\frac{10^5 s_1 D_n}{2EA} =$ $10^5 N D_n$	$(9) \times (11)$ G	$\frac{w \times (11)}{10^5}$ H	$(10) - (12) - 1$ $F - G - 1$	From chart $\frac{x}{c_n}$	$(4) - (15)$ c_n	cosh (15) $\cosh \frac{x}{c_n}$	$w(16)(17)$ $T =$ $wc_n \cosh \frac{x}{c_n}$	$[(18) - w] - (16)$ sag $=$ $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1 000265	0 94755	0 0013285	0 00001460	−0 001063	0 0250	4,761 6	1 00031	7,339 9	1 5
1 000613	1 42036	0 0013281	0 00002189	−0 000715	0 0429	4,158 0	1 00092	6,413 4	3 8
1.001069	1.89207	0 0013275	0 00002916	−0 000259	0 0633	3,752 1	1 00200	5,793 6	7 5
1 001667	2 36226	0 0013267	0 00003640	0 000350	0 0850	3,486 6	1 00361	5,392 2	12 3
1 002401	2 83058	0 0013257	0 00004362	0 001075	0 1066	3,328 8	1.00568	5,158 8	18 9
1 003269	3 29667	0 0013246	0 00005080	0 001944	0 1286	3,210 9	1 00828	4,989 0	26 6
1 004272	3 76017	0 0013233	0 00005794	0 002949	0 1491	3,155 7	1 01114	4,917 1	35 1
1 005410	4 22074	0 0013219	0 00006504	0 004088	0 1708	3,088 7	1 01462	4,829 3	45 2
1 006680	4 67805	0 0013203	0 00007209	0 005360	0 1913	3,052 6	1 01836	4,790 4	56 0

TABLE 16.—CATENARY AT 30°F.
Cable without Ice or Wind Load

$w = 1\ 5410$
$t_n = 30°\ F$
$D_n = 1 + 9\ 22 \times 10^{-6}(30 + 10)$
$= 1\ 00036888$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
$(6) \times D_n$ F	$(8) \times D_n$ $\frac{10^6 s_1 D_n}{2EA} =$ $10^6 N D_n$	$(9) \times (11)$ G	$\frac{w \times (11)}{10^6}$ H	$(10) - (12) - 1$ $F - G - 1$	From chart $\frac{x}{c_n}$	$(4) \div (15)$ c_n	cosh (15) $\cosh \frac{x}{c_n}$	$w(16)(17)$ $T =$ $wc_n \cosh \frac{x}{c_n}$	$[(18) - w] - (16)$ sag $=$ $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1 000634	0 94790	0 0013290	0 00001461	−0 000695	0 0335	3,553 5	1 00056	5,479 0	2 0
1 000982	1 42088	0 0013286	0 00002190	−0 000317	0 0533	3,346 7	1 00142	5,161 6	4 8
1 001438	1 89277	0 0013280	0 00002917	0 000110	0 0736	3,227 1	1 00271	4,986 4	8 6
1.002036	2,36313	0 0013272	0 00003642	0 000709	0 0941	3,149 4	1 00443	4,874 7	13 9
1 002771	2 83162	0 0013262	0 00004363	0 001445	0 1146	3,096 4	1 00657	4,802 9	20 3
1 003639	3 29789	0 0013251	0 00005082	0 002314	0 1351	3,056 4	1 00914	4,753 0	28 0
1.004642	3 76156	0 0013238	0 00005797	0 003318	0 1559	3,018 0	1 01218	4,707 4	36 8
1 005781	4 22230	0 0013223	0 00006507	0 004459	0 1768	2,983 8	1 01567	4,670 1	46 8
1.007051	4.67978	0.0013207	0 00007212	0 005730	0 1967	2,968 8	1 01940	4,663 7	57 6

TABLE 17 — CATENARY AT 60°F.
Cable without Ice or Wind Load

$w = 1\ 5410$
$t_n = 60°\ \text{F}$
$D_n = 1 + 9\ 22 \times 10^{-6}(60 + 30)$
$= 1\ 0006454$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
$(6) \times D_n$ F	$(8) \times D_n$ $\frac{10^6 s_1 D_n}{2EA} =$ $10^6 N D_n$	$(9) \times (11)$ G	$\frac{w \times (11)}{10^6}$ H	$(10) - (12) - 1$ $F - G - 1$	From chart $\frac{x}{c_n}$	$(4) \div (15)$ c_n	cosh (15) $\cosh \frac{x}{c_n}$	$w(16)(17)$ $T =$ $wc_n \cosh \frac{x}{c_n}$	$[(18) - w] - (16)$ sag $=$ $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1 000911	0 948161	0 0013294	0 00001461	−0 000418	0 0416	2,861 6	1 00087	4,413 6	2 5
1,001259	1 42128	0 0013289	0 00002190	−0 000070	0 0620	2,877 1	1 00192	4,442 1	5 5
1 001715	1 89329	0 0013283	0 00002918	0 000387	0 0814	2,917 8	1 00331	4,511 2	9 7
1 002314	2 36378	0 0013275	0 00003643	0 000986	0 1012	2,928 4	1 00523	4,536 2	15 3
1 003048	2,83241	0 0013266	0 00004365	0 001721	0 1205	2,944 8	1 00727	4,570 9	21 4
1 003917	3,29880	0 0013255	0 00005083	0 002591	0 1408	2,932 7	1 00993	4,561 2	29 1
1 004920	3 76260	0 0013242	0 00005798	0 003596	0 1607	2,927 0	1 01294	4,570 3	37 9
1 006059	4,22346	0 0013227	0 00006508	0 004736	0 1809	2,916 2	1 01640	4,567 6	47 8
1 007330	4 68107	0 0013211	0 00007213	0 006009	0 2008	2,908 2	1 02023	4,572 2	58 8

TABLE 18.—CATENARY AT 100°F.
Cable without Ice or Wind Load

$w = 1\ 5410$
$t_n = 100°\ F$
$D_n = 1 + 9\ 22 \times 10^{-6}\ (100 + 10)$
$= 1\ 001011$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
(6) × D_n F	(8) × D_n $\frac{10^6 s_1 D_n}{2EA} =$ $10^6 N D_n$	(9) × (11) G	$\frac{w \times (11)}{10^6}$ H	(10) − (12) − 1 $F - G - 1$	From chart $\frac{x}{c_n}$	(1) ÷ (15) c_n	cosh (15) $\cosh \frac{x}{c_n}$	$w(16)(17)$ $T =$ $wc_n \cosh \frac{x}{c_n}$	[(18) − w] − (16) sag = $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1.001275	0 94851	0 0013299	0 00001462	−0 000055	0 0510	2,204 5	1 00146	3,402 1	3 2
1.001624	1.42180	0 0013294	0 00002191	0 000295	0 0731	2,440 2	1 00267	3,770 3	6 5
1 002080	1 89399	0 0013288	0 00002919	0 000751	0 0913	2,601 4	1 00417	4,025 5	10 9
1 002679	2 36466	0 0013280	0 00003644	0 001351	0 1099	2,696.6	1.00605	4,180 6	16 3
1 003413	2.83345	0 0013271	0 00004366	0 002086	0 1282	2,767 9	1 00823	4,300 4	22 8
1,004282	3 30001	0 0013259	0 00005085	0 002956	0 1477	2,795 7	1 01092	4,355 2	30 5
1,005286	3.76398	0 0013246	0 00005800	0 003961	0 1667	2,822 5	1 01392	4,410 0	39 3
1,006426	4 22502	0 0013232	0 00006511	0 005103	0 1857	2,840 8	1 01729	4,453 4	49 1
1,007697	4.68279	0.0013216	0 00007216	0 006375	0 2058	2,837 5	1 02125	4,465 5	60 3

This sag is usually not of particular interest, since it does not represent a critical condition, except in the case of supports at different elevations. In this case the minimum sag may have to be investigated to determine whether uplift on the lower support may not occur at minimum temperature.

The remaining columns (7), (8) and (9) of Table 13 are constants of Eq. (542), which are used in the calculation of the remaining curves, and are tabulated here for convenience.

From the data already computed and tabulated in Tables 12 and 13 and with the use of Eq. (542), the remaining curves are all easily calculated. The method is exactly the same for all curves. It will therefore be sufficient to illustrate this method as applied to the data of Table 14

By subtracting 1 from each member of Eq. (542), which is the abbreviated form of Eq. (541), this equation becomes

$$\frac{\sinh \frac{x}{c_n}}{\frac{x}{c_n}} - 1 = F_n - G_n - 1 + H_n\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right].$$

From Eqs. (543) to (545), inclusive,

$$D_n = 1 + \alpha(t_n - t_1)$$

$$N = \frac{s_1}{2EA}$$

$$P_1 = \left[\frac{\cosh \frac{x}{c_1}}{\frac{x}{c_1}} + \frac{1}{\sinh \frac{x}{c_1}}\right].$$

Whence, by Eqs. (546), (547) and (548),

$$F_n = \frac{s_1 D_n}{x}$$

$$G_n = w_1 P_1 N D_n$$

$$H_n = w_n N D_n.$$

From these equations, columns (7), (8) and (9) of Table 13 and columns (10), (11), (12), (13) and (14) of each of the other tables, representing catenaries for various loadings and temperatures, are readily computed. Column (15) in each case is obtained from the chart by the method already explained under the heading, Method of Solving Eq. (541) for $x \div c_n$ Once the new arguments $x \div c_n$ have been found, columns (16), (17), (18) and (19) follow from the relations given below:

$$c_n = x \div (x \div c_n) \tag{16}$$

$$\cosh \frac{x}{c_n} \text{ obtained from tables of hyperbolic cosines} \tag{17}$$

$$T = w_n c_n \cosh \frac{x}{c_n} \tag{18}$$

where w_n is the particular total loading, per unit length of cable, appropriate to the catenary in question.

$$\text{Sag} = c_n\left(\cosh \frac{x}{c_n} - 1\right). \qquad (19)$$
$$= \frac{T}{w_n} - c_n$$

The curves of Fig. 70 are the graphs corresponding to the computed values set down in columns 18 and 19 of the above mentioned tables.

PROBLEMS

1. A 0000 medium hard-drawn, copper cable is to be strung on poles along a level right-of-way. The points of attachments of conductors to insulators are 42 ft. above ground, and the minimum permissible clearance of conductors to ground is 30 ft. The conductor tension at the supports under stringing conditions must not exceed 1,400 lb. What is the maximum allowable distance between supports? What is the length of cable between supports?

2. The line of Problem 1 crosses an inlet. At the place of crossing the minimum available span is 800 ft. The minimum allowable clearance between cable and water is 40 ft. Allowing the same tension as before, what is the required height of the point of conductor attachment above water level? What is the sag at the middle of the span? What is the length of cable in the span?

3. Calculate and plot the temperature-tension curve and the sag curve at 100° F., for a 250,000-cir. mil, stranded, copper cable for which the following data apply. Make calculations covering spans up to 800 ft. for a level right-of-way.

Cable diameter = 0.575 in.

Weight of cable per foot = 0.772 lb.

Ice load assumed at maximum loading = 0.5 in. all around cable.

Minimum temperature at which ice load exists = −10° F.

At maximum load and minimum temperature of −10° F., the wind load, assumed at right angles to line, and acting on ice-covered cable, is 10 lb. per square foot of projected area.

Maximum allowable tension in cable = 4,200 lb.

Modulus of elasticity E = 12,000,000 per square inch.

Coefficient of linear expansion = 9.22×10^{-6}.

CHAPTER XI

MECHANICAL DESIGN

SPANS WITH SUPPORTS AT UNEQUAL ELEVATIONS

Spans with Supports at Unequal Elevations.—Where the transmission-line right-of-way passes through very hilly or mountainous country, the length of span is frequently determined more by the limited choice of possible tower locations than by the conditions of maximum economy, which are discussed in a succeeding chapter. The roughness of the country and the greatly restricted number of suitable tower sites make it frequently necessary to construct spans in which there is a considerable difference in elevation of the points of attachment of a given conductor on adjacent towers. In general, the method followed in the design of such spans is similar to that followed for spans with points of conductor attachments at equal elevations. In addition, however, the designer must be assured that under conditions of minimum sag the conductors do not exert an upward pull upon the towers. The tension in the conductors will be unequal at adjacent towers, as will also the vertical distances from the points of support to the lowest point on the suspended conductors. A simple method of finding these required quantities will be outlined.[1]

Theory Outlined.—In Fig. 71, let AD_0B_0 be a catenary representing a conductor supported at A and at B_0, two points whose elevations differ by b_0 units, and let the length of span be a The conductor is to be strung so that (a) when under maximum loading and minimum temperature conditions the tension at A will not exceed the maximum allowable tension assumed for the conductor; (b) when under minimum sag the conductor will not exert an upward pull on the tower at B_0. This condition is

[1] Smith, G. S., "Transmission Line Design. Mechanical Design of Spans with Supports at Unequal Elevations," presented before the A. I. E. E. Pacific Coast Convention 1925; also University of Washington Engineering Experiment Station, *Bull.* 29.

fulfilled by the stipulation that the lowest point D_0, on the curve, must at all times lie between B_0 and A.

In Fig. 71 if the catenary AD_0B_0 be extended to the point C_0, having the same elevation as A, a catenary is obtained which is symmetrical with respect to the Y-axis through the lowest point D_0. For this curve, the half span length is x_0 and the maximum deflection is y_0. All quantities, such as tension, deflection and length for the symmetrical curve, may be computed by the methods already described for spans with supports at equal elevations. Remembering that distances a, x and x_0 etc., are measured horizontally between supports, it is apparent that at any point on the curve distant x units from D_0, the point of maximum sag on the symmetrical curve, the deflection is given by Eq. (488), as

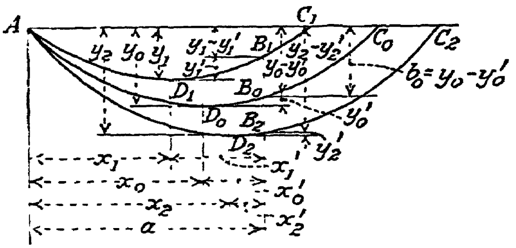

FIG. 71.—Catenaries for spans with supports at unequal elevations.

$$y = c\left(\cosh \frac{x}{c} - 1\right)$$

where the constant c, for any given conditions of temperature and loading, is the ratio $H \div w$ for that set of conditions. If the half span length x_0, of the symmetrical curve AD_0C_0, were known, the distance x_0', from the point of maximum deflection to the nearest support, could readily be found, for

$$x_0' = a - x_0. \tag{549}$$

Once x_0' is known, the maximum deflection y_0 of a symmetrical catenary whose span length is $2x_0$, is available from the equation

$$y_0' = c_0\left(\cosh \frac{x_0'}{c_0} - 1\right) \tag{550}$$

and since

$$b_0 = y_0 - y_0' \tag{551}$$

the difference in elevation of the two supports is expressible in terms of the two symmetrical catenaries having the span lengths $2x_0$ and $2x_0'$. The problem, then, is to find these span lengths.

Interpolation for x_0.—The values x_0 and x_0' of the half span lengths for the long and short spans of the corresponding symme-

trical catenaries may readily be found by interpolation from plotted values of the relation

$$b = y - y'$$

if the range of arguments is so chosen as to embrace the argument corresponding to the desired value $b - b_0$ The following paragraphs will make the truth of this statement more apparent.

Consider a large number of catenaries similar to the one already described, some having spans longer and some shorter than $2x_0$. Let the conductors representing these curves be all alike, let them all be strung to have the same maximum tension under conditions of minimum temperature and maximum loading, and assume that all of them have undergone the same changes of loading and temperature.

Of this series of catenaries AD_1C_1 is one that is a small amount shorter, and AD_2C_2 one a small amount longer than AD_0C_0 For these curves the maximum sags are

$$\left.\begin{aligned} y_1 &= c_1\left(\cosh \frac{x_1}{c_1} - 1\right) \\ y_2 &= c_2\left(\cosh \frac{x_2}{c_2} - 1\right) \end{aligned}\right\} \qquad (552)$$

for the symmetrical catenaries of span lengths $2x_1$ and $2x_2$, respectively, and

$$\left.\begin{aligned} y_1' &= c_1\left(\cosh \frac{x_1'}{c_1} - 1\right) \\ y_2' &= c_2\left(\cosh \frac{x_2'}{c_2} - 1\right) \end{aligned}\right\}$$

for the symmetrical catenaries of span lengths $2x_1'$ and $2x_2'$, respectively, by Eq. (488). The above values of half span lengths are related as in Eq. (549); that is,

$$\left.\begin{aligned} x_1' &= a - x_1 \\ x_2' &= a - x_2 \end{aligned}\right\} \qquad (553)$$

Since B_1 and B_2 are the points of attachment, the differences in elevation between points of attachment on the two supports for the conductor AD_1C_1 and AD_2C_2 are, respectively,

$$b_1 = y_1 - y_1$$

and

$$b_2 = y_2 - y_2'.$$

An inspection of the Fig. 70 clearly shows that

$$b_2 > b_0 > b_1$$

and hence it follows that, by assuming a number of arguments $\frac{x}{c}$ and computing the corresponding values of b from Eq. (551), the value of b_0 may be found by interpolation if the series of arguments chosen includes values both larger and smaller than $\frac{x_0}{c_0}$

The differences b are plotted against their corresponding values of T_m, c or x, and the values of x_1, c_0 and T_0, etc , corresponding to the curve AD_0B_0, readily follow. The method described will be illustrated by an example.

Example.—A 500,000-cir mil copper cable with the constants and subjected to the loadings and temperature conditions described in the example on page 205, is suspended from supports 600 ft. apart The difference in elevations of the points of suspension is 40 ft. It is required (*a*) to find the tension at the highest support for each temperature and loading; (*b*) to find the corresponding maximum sags, and (*c*) to test the results of the solution to see whether under minimum sag conditions there is an upward pull on the lowest support. (If an upward pull exists the maximum tension may be reduced or the span may be redesigned for a smaller difference in elevation between supports)

Solution.—The given data and working equations are

$$a = 600 \text{ ft.} \qquad x' = a - x$$

$$b_0 = 40 \text{ ft.} \qquad y' = c\left(\cosh\frac{x'}{c} - 1\right)$$

The symmetrical spans (spans with supports at equal elevations), have already been computed for lengths up to nearly 1,200 ft. for this particular conductor. The results of these calculations are given in Tables 12 to 18 inclusive. From these tables the already known values of x_1, c and y of Eq. (552) are copied and set down in columns (21), (22) and (23) of Table 19, for the corresponding values of $\frac{x}{c_1}$. From Eq. (553) the values of x' in column (24) follow. After computing columns (25) and (26), the values, of y' in column (27) and b in column (28), are readily calculated These values of b are next plotted against x, c, T and y of columns (4), (16), (18) and (19), respectively, and the values corresponding to $b_0 = 40$ ft are read off and tabulated as in Table 20.

It will be observed that the vertical sag is at no time less than 40 ft., and hence no upward pull will ever be exerted on the support.

Stringing Conductors.—If the cable is to be strung by the use of a dynamometer, the tensions of Table 20 will furnish the required

TABLE 19.—SUPPORTS AT UNEQUAL ELEVATIONS
Span = 600 ft.
b_0 = 40 ft.

Temperature and load	(20) Assumed values from (1) $\frac{x}{c_1}$	(21) From (16) c_n	(22) From (19) y	(23) From (4) a	(24) 600 − (23) $x' = a - x$	(25) (24) − (21) $\frac{x'}{c_n}$	(26) cosh (25) $\cosh \frac{x'}{c_n}$	(27) (21)[(26) − 1] $y' = c_n (\cosh \frac{x'}{c_n} - 1)$	(28) (22) − (27) $b = y - y'$
32° F., cable loaded	0 12	2,783 1	22 7[1]	354 8	245 2	0 08810	1 00388	10 8[1]	10 0[2]
	0 14	2,818 6	30 3	412 9	187 1	0 06638	1 00221	6 2	20 3
	0 16	2,834 4	39 1	470 5	129 5	0 04560	1 00104	2 9	30 5
	0 18	2,839 3	49 2	527 5	72 5	0 02553	1 00033	0 9	40 7
	0 20	2,847 2	60 1	584 0	16 0	0 00562	1 00002	0 1	50 6
−10° F., cable only	0 12	3,328 8	18 9	354 8	245 2	0 07366	1 00272	9 1	9 8
	0 14	3,210 9	26 6	412 9	187 1	0 05827	1 00170	5 5	21 1
	0 16	3,155 7	35 1	470 5	129 5	0 04104	1 00084	2 7	32.4
	0 18	3,088 7	45 2	527 5	72 5	0 02347	1 00028	0 9	44 3
	0 20	3,052 6	56 0	584 0	16 0	0 00524	1 00001	0 0	56 0
30° F., cable only	0 12	3,096 4	20 3	354 8	245 2	0 07919	1 00313	9 7	10 6
	0 14	3,056 4	28 0	112 9	187 1	0 06122	1 00187	5 7	22 3
	0 16	3,018 0	36 8	470 5	129 5	0 04291	1 00092	2 8	34 0
	0 18	2,983 8	46 8	527 5	72 5	0 02430	1 00030	0 9	45 9
	0 20	2,968 8	57 6	584 0	16 0	0 00539	1 00001	0 0	57 6
60° F., cable only	0 12	2,911 8	21 4	354 8	245 2	0 08327	1 00317	10 2	11 2
	0 14	2,932 7	29 1	412 9	187 1	0 06380	1 00204	6 0	23 1
	0 16	2,927 9	37 9	470 5	129 5	0 04423	1 00098	2 9	35.0
	0 18	2,916 2	47 8	527 5	72 5	0 02486	1 00031	0 9	46 9
	0 20	2,908 2	58 8	584 0	16 0	0 00550	1 00002	0 1	58 7
100° F., cable only	0 12	2,767 9	22 8	354 8	245 2	0 08859	1 00393	10 9	11 9
	0.14	2,795 7	30 5	412 9	187 1	0 06692	1 00221	6 3	24.2
	0 16	2,822 5	39 3	170 5	129 5	0 04588	1 00105	3 0	36 3
	0 18	2,810 8	49 1	527 5	72 5	0 02552	1 00093	0 9	48 2
	0.20	2,837 5	60 3	584 0	16 0	0 00561	1 00002	0 1	60 2

[1] For the loaded cable at 32° F., the values of y and y^1 represent sags in the plane of the resultant force
[2] For the loaded cable at 32° F., the values in the table are computed values of $(y - y')$ multiplied by $(\cos \theta = 0.8127)$.

stringing data. On the other hand, if sag is to be the basic criterion, two methods are open: (a) The cable may be suspended from its insulators and allowed to sag until the distance AA'' of Fig. 72 is equal to the value of y_0 in column (28) of Table 19, for the temperature at the time of stringing. (b) The cable may be sagged until the distance $AA' = BB_0'$ is such that the line of sight $A'B_0$ (parallel to AB_0) is tangent to the cable. The point D' is located at the intersection of a vertical line, midway between supports, with the catenary, and is taken as the lowest point on the curve. This is not strictly accurate, but the error made by this assumption is so small as to be entirely negligible here. The value of AA' may therefore be easily calculated for

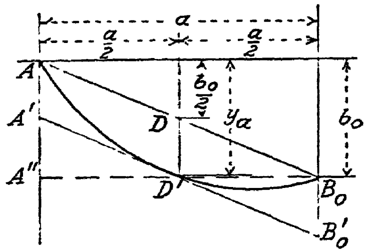

FIG. 72 —Determination of stringing sag

$$y_a = c\left(\cosh\frac{a}{2c} - 1\right)$$

whence

$$AA' = y_a - \frac{b_0}{2}. \tag{554}$$

TABLE 20

Values of x, c, T_m and y, corresponding to $b_0 = 40$ ft., are obtained from plotted curves using b of column (28) as the independent variable, and each of the quantities sought as the dependent variable.

Temperature and load	x_0	c_0	T_0	y_0
32° F., cable loaded	523.8	2,838.9	8,094	40 8
−10° F., cable only. .	506 9	3,109 2	4,852	41 4
30° F., cable only.. .	499 4	3,004 8	4,692	41 8
60° F , cable only... .. .	494 4	2,922 7	4,568	41 9
100° F., cable only	488 0	2,831 6	4,428	42 3

PROBLEM

The difference in elevation between two adjacent supports in a 700-ft. span, of the line in Problem 3, Chap. X, is 45 ft. It is assumed that the tension at the highest support under maximum loading conditions, the snow load, wind load and all other pertinent data, are as given in the problem above mentioned. Calculate, for a temperature of 110° F., (a) the tension at the highest support; (b) the sag y_0; (c) the length of cable in the span.

CHAPTER XII

ECONOMICS OF SPAN DESIGN

The Most Economical Tower Spacing.—In the design of important lines, and particularly for lines or sections of lines built over more or less level country where spans of uniform length are possible, it is essential that the designer make a careful investigation to determine the length of span which will require minimum total capital outlay. In very rough country where tower locations and lengths of spans are frequently determined within narrow limits by the nature of the terrain, the principles of maximum economy, here discussed, will naturally be applied with greater difficulty. As will be observed from a close study of the problem, the span length of greatest economy is not a sharply defined one on either side of which costs rise rapidly; but, rather. there is a considerable variation in span length over which total costs change rather slowly. Therefore, if so required to suit the contour of the right of way, the span may be varied considerably in length from the one found to be most economical, without seriously affecting the cost.

Economic Principle.—The principles underlying the problem of finding the most economical tower spacing may be stated as follows: It is assumed, as is usually the case, that the minimum permissible clearance between conductor and ground is fixed, either by state law or otherwise. This is the fundamental datum. Furthermore, in any span the minimum clearance occurs when the deflection is maximum. The latter condition, in turn, is defined by the critical catenary.

Thus, with a fixed ground clearance to begin with, the height of the individual support increases with the length of span, whereas their total number decreases. The total cost of the supports is equal to the number of units required times the cost of a unit. With increasing length of span, therefore, the cost of the individual unit is increased on account of its greater height and the greater strength required, while the number of units is, of course, diminished. The most economical spacing to use is the one for

which the total cost of supports is a minimum; that is, it is a spacing such that, if increased by a small increment, the total increased cost for all support, due to the increased cost of a unit, will be exactly equalled by the total reduction in cost resulting from their decreased number.

Methods of Attack.—Cut and try methods are frequently followed in estimating the most economical tower spacing When this method is used the costs of the several separate items which go to make up the total cost of the unit, such as insulators, foundations, towers, cost of erection, etc., are separately estimated for each of a number of assumed lengths of span. Curves for each of these variable cost items are then drawn with cost expressed as a function of span length. A total cost curve, made up by adding the ordinates to the separate curves, is then constructed. The minimum point on this curve locates the most economical length of span.

In the analysis which follows, instead of the cut-and-try method, a mathematical solution of the problem is offered. This method has the advantage of ease of manipulation, and of furnishing results which may readily be incorporated into a mathematical study of the greater problem having to do with the economics of design involving the line as a whole, as considered in a subsequent chapter.

Basis for Mathematical Solution.[1]—The method by which one may proceed to a mathematical statement of the problem is suggested by the discussion under the caption entitled "Economic Principle," together with a consideration of cost items that enter into the total cost of the supports and how they vary.

The total cost of a tower structure may readily be segregated into various items of cost. For the purpose in hand, the following list of items is used:

1. Cost of tower at place of erection
2. Cost of erection of tower
3. Cost of lease or purchase of tower site
4. Cost of foundation installed
5. Cost of location and inspection of support
6. Cost of insulators at tower location
7. Cost of placing insulators and cable.

In order to make a solution for the most economical tower spacing it is desirable to separate the above items of cost into two

[1] KIRSTEN, F. K. University of Washington Engineering Experiment Station, *Bull.* 17.

groups, one of which is a function of the tower spacing only, and the other of which varies both with the tower spacing and the height of the tower. The total cost will then be expressed in mathematical form in terms of these two groups of cost items. The first question to be answered is, into which class does each of the above items fall?

Item 1.—The weight of the tower is proportional to the maximum cable stress for which it is designed. For a given stress, an increase in the height of the tower increases its weight in proportion to the square of its height. Since the cost of the tower is proportional to its weight, the amount of item 1 is in direct proportion to the square of the height of the tower

Item 2.—The given stress for which the tower is designed practically fixes the weight per unit length and the lengths of its structural members. Hence an increase in the height of the tower increases the number of its structural members, and, consequently, the cost of erection, in direct proportion to the square of its height.

Item 3 —Since in practical tower design the ratio of the height of the tower to the width of its base is a constant, the area of the tower site and therefore the cost of its lease or purchase will also vary in direct proportion to the square of the height of the tower. (Where an entire right-of-way is purchased, this item should be omitted.)

Item 4.—The cost of the foundation is directly proportional to the tension for which the tower is designed and is practically independent of the height of the tower.

Items 5, 6 and 7.—These items are independent of the mechanical features of the towers and are constants for given transportation rates and market conditions of materials and labor. The magnitude of item 6 is practically proportional to the transmission voltage and is fixed for a given transmission voltage.

Equations Derived.—Let

L = length of the transmission line
$2x$ = tower spacing
h = height of tower
k_1 = minimum clearance of cable to ground
y = maximum deflection of the catenary formed by the cable.

The number of line supports is, then,

$$\text{Number} = \frac{L}{2x} \tag{555}$$

and the height of the support is

$$h = k_1 + y \tag{556}$$

The total cost of the line supports is

$$\text{Cost} = \text{No. } (h^2k_2 + k_3) \tag{557}$$

where

$$h^2k_2 = \text{sum of items 1, 2 and 3}$$
$$k_3 = \text{sum of items 4, 5, 6 and 7.}$$

Substituting Eqs. (555), and (556) in Eq. (557),

$$\text{Cost} = \frac{L}{2x}[k_2(k_1 + y)^2 + k_3]. \tag{558}$$

Substituting for y in the above equation its equivalent from Eq. (488),

$$\text{Cost} = \frac{L}{2x}\left[k_2\left\{k_1 + c\left(\cosh\frac{x}{c} - 1\right)\right\}^2 + k_3\right]. \tag{559}$$

For a minimum or maximum cost the derivative of Eq. (559) with respect to x must be equal to zero:

$$\frac{d(\text{Cost})}{dx} = 0 = \frac{L}{2}\left(\frac{2k_2c\cosh\frac{x}{c}\sinh\frac{x}{c}}{x} - \frac{k_2c^2\cosh^2\frac{x}{c}}{x^2} - \frac{2k_2c\sinh\frac{x}{c}}{x} + \frac{2k_2c^2\cosh\frac{x}{c}}{x^2} - \frac{k_2c^2}{x^2} + \frac{2k_1k_2\sinh\frac{x}{c}}{x} - \frac{2k_1k_2c\cosh\frac{x}{c}}{x^2} + \frac{2k_1k_2c}{x^2} - \frac{k_1^2k_2}{x^2} - \frac{k_3}{x^2}\right)$$

$$= 2cx\cosh\frac{x}{c}\sinh\frac{x}{c} - c^2\cosh^2\frac{x}{c} - 2cx\sinh\frac{x}{c} + 2c^2\cosh\frac{x}{c} - c^2 + 2k_1x\sinh\frac{x}{c} - 2k_1c\cosh\frac{x}{c} + 2k_1c - k_1^2 - \frac{k_3}{k_2}. \tag{560}$$

But $x = c \times \frac{x}{c}$, hence

$$0 = c^2\left(2\frac{x}{c}\cosh\frac{x}{c}\sinh\frac{x}{c} - \cosh^2\frac{x}{c} - 2\frac{x}{c}\sinh\frac{x}{c} + 2\cosh\frac{x}{c} - 1\right) + 2ck_1\left(\frac{x}{c}\sinh\frac{x}{c} - \cosh\frac{x}{c} + 1\right) - k_1^2 - \frac{k_3}{k_2}. \tag{561}$$

Simplifying,

$$c^2\left(\cosh\frac{x}{c}-1\right)\left(2\frac{x}{c}\sinh\frac{x}{c}-\cosh\frac{x}{c}+1\right) + 2ck_1\left(\frac{x}{c}\sinh\frac{x}{c}-\cosh\frac{x}{c}+1\right) = k_1^2+\frac{k_3}{k_2} \quad (562)$$

from which

$$c = -\frac{k_1\left(\frac{x}{c}\sinh\frac{x}{c}-\cosh\frac{x}{c}+1\right)}{\left(\cosh\frac{x}{c}-1\right)\left(2\frac{x}{c}\sinh\frac{x}{c}-\cosh\frac{x}{c}+1\right)} \pm \left[\frac{\left(k_1^2+\frac{k_3}{k_2}\right)\left(\cosh\frac{x}{c}-1\right)\left(2\frac{x}{c}\sinh\frac{x}{c}-\cosh\frac{x}{c}+1\right) + k_1^2\left(\frac{x}{c}\sinh\frac{x}{c}-\cosh\frac{x}{c}+1\right)^2}{\left(\cosh\frac{x}{c}-1\right)^2\left(2\frac{x}{c}\sinh\frac{x}{c}-\cosh\frac{x}{c}+1\right)^2}\right]^{\frac{1}{2}} \quad (563)$$

Simplifying Eq. (563),

$$c = -\frac{k_1\left[\frac{x}{c}\sinh\frac{x}{c}-\left(\cosh\frac{x}{c}-1\right)\right]}{\left(\cosh\frac{x}{c}-1\right)\left[2\frac{x}{c}\sinh\frac{x}{c}-\left(\cosh\frac{x}{c}-1\right)\right]} \pm \frac{\left\{\frac{k_3}{k_2}\left(\cosh\frac{x}{c}-1\right)\left[2\frac{x}{c}\sinh\frac{x}{c}-\left(\cosh\frac{x}{c}-1\right)\right] + \left(k_1\frac{x}{c}\sinh\frac{x}{c}\right)^2\right\}^{\frac{1}{2}}}{\left(\cosh\frac{x}{c}-1\right)\left[2\frac{x}{c}\sinh\frac{x}{c}-\left(\cosh\frac{x}{c}-1\right)\right]} \quad (564)$$

For a given clearance k_1 of the cable to ground, and a given hyperbolic argument, Eq. (564) contains only one variable, the ratio $k_3 \div k_2$. In practical line design the possible range of this ratio usually falls within the limits 500 and 2,500.

The Most Economical Span S as a Function of Conductor Diameter d_s.—From Eq. (564) the minimum cost of tower may be found if the positive sign before the radical is used. For a given ground clearance k_1 and a given ratio $k_3 \div k_2$, this equation yields the value of c for any assumed argument $\frac{x}{c}$. Since c is

given as a function of $\frac{x}{c}$ it follows that the corresponding tower spacing $2x = S$ is also known as a function of c. Thus, from Eq. (564), curves may be plotted giving values of c as a function of $S = 2x$, for any assumed values of ground clearances k_1 and any ratios $k_3 \div k_2$. For each ratio $k_3 \div k_2$ chosen, there is a family of curves; an individual curve results for each assumed value of k_1. Such a set of curves, applying to stranded, copper cables, is illustrated in the graphs marked 1, 2, 3, etc , of Fig 73. By assuming values of k_1 and $k_3 \div k_2$ of a range sufficient to cover any conditions likely to be encountered in practice, a single set of curves will suffice for any problem likely to arise. In these curves it must be remembered that $2x$ represents the *most economical span* for the corresponding values of c, k_1, and $k_3 \div k_2$.

If now, on these curves, there be superimposed another set of curves relating the values of c and $2x$, pertaining to the critical catenary for each conductor size of a given conductor material, a ready means is provided for finding the most economical span corresponding to any conductor size and for the material represented. The intersection points of the two sets of curves determine the spans of maximum economy for the various conditions and conductor sizes. The values of c and $2x$ for the critical catenary (the catenary at 100° F. is taken as the critical condition in the work here discussed), are computed by the method already discussed in the previous chapter and there illustrated for a 500,000-cir. mil, copper cable.

Since the circular mil area of a cable is expressible as a function of the overall cable diameter; that is,

$$\begin{aligned} \text{cir. mil} &= d^2 \\ &= d_s^2 \div (1.151)^2 \end{aligned}$$

where the constant 1.151 represents approximately the ratio of outside diameter to the diameter of the equivalent solid rod for stranded cables in sizes above 250,000 cir. mils, the second set of curves of Fig. 73 may be indexed in terms of cable diameters, and the intersections of the two sets of curves may be used to evaluate the most economical span as a function of d_s. That is, by plotting the corresponding values of S and d_s obtained from the intersections, a graphical relation between conductor diameter and most economical span is obtained. This relation is then reduced

to mathematical form by writing an empirical equation to fit the curves over the range covered by the investigation.

As a basis for much of the work of the chapter, computations were made from the results of which a large number of curves

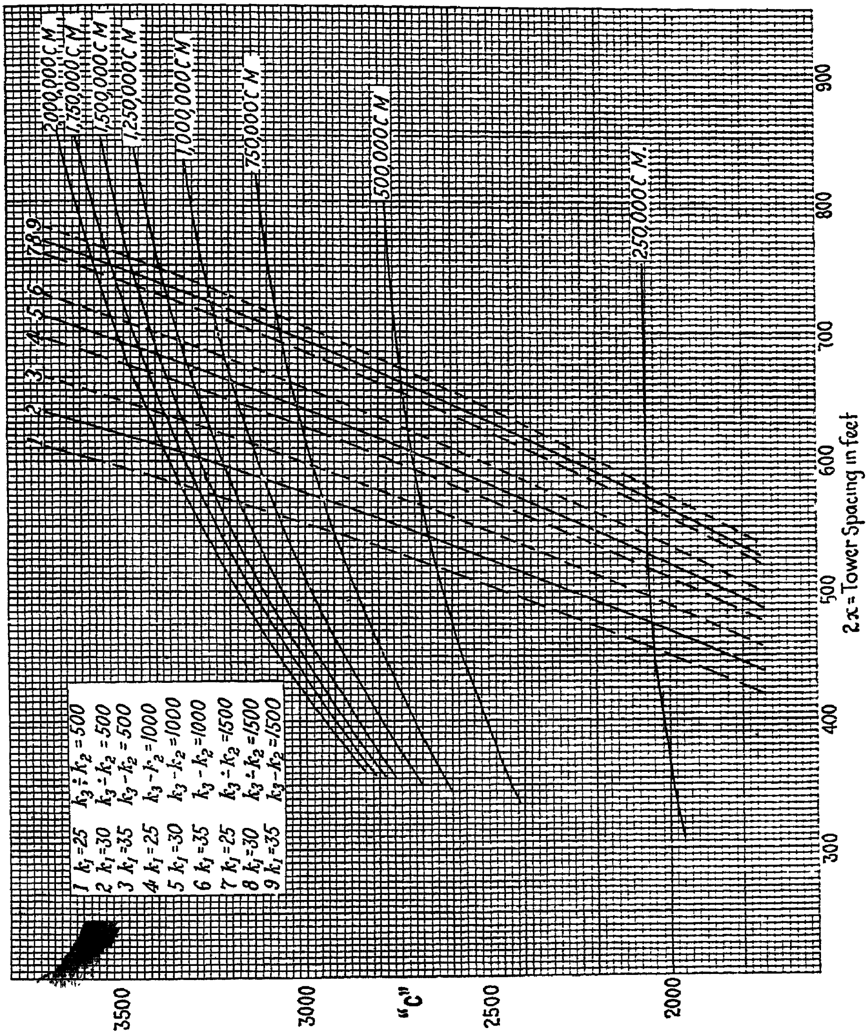

FIG. 73.—Curves for finding the most economical span S.

similar to those of Fig. 73 were drawn. The computations covered the three conductor materials, aluminum, copper and steel for cable sizes between 250,000 and 2,000,000 cir. mils, inclusive. The calculations were repeated for ground clearance

tion will still fit the curve closely for values below 0.6 and above 2, but additional constants k_6, k_7 and k_8 are required.

The Most Economical Tower Height.—A study of the maximum sags, for the critical catenaries, corresponding to the spans

TABLE 21.—MAXIMUM CATENARY DEFLECTIONS (IN FEET) FOR MOST ECONOMICAL TOWER HEIGHT

$\frac{k_3}{k_2}$	Conductor material	Conductor diameter d_s: 0 999	1 153	1 289	1 412	1 525	1 631	Average cable deflection	
500	Al	12 70	12 90	12 72	12 73	12 72	12 70	12 69	$k_1 = 25$
	Cu	12 70	12 72	12 73	12 69	12 66	12 70		
	Fe	12 78	12 68	12 65	12 60	12 54	12 58		
1,000	Al	16 32	16 43	16 32	16 41	16 36	16 33	16 34	
	Cu	16 33	16 35	16 40	16 40	16 40	16 42		
	Fe	16 35	16 36	16 32	16 28	16 25	16 29		
1,500	Al	19 50	19 52	19 49	19 54	19 52	19 47	19 54	
	Cu	19 45	19 51	19 47	19 53	19 58	19 54		
	Fe	19 47	19 60	19 65	19 60	19 61	19 62		
2,000	Al	22 37	22 31	22 33	22 42	22 38	22 34	22 38	
	Cu	22 38	22 35	22 39	22 40	22 37	22 37		
	Fe	22 34	22 42	22 44	22 45	22 42	22 45		
2,500	Al	24 95	24 93	24 98	24 95	24 98	24 96	24 98	
	Cu	25 00	25 00	24 98	24 97	24 99	25 03		
	Fe	24 96	24 96	25 02	25 04	25 04	26 06		
500	Al	13 73	13 96	13 77	13 78	13 75	13 78	13 74	$k_1 = 30$
	Cu	13 75	13 79	13 82	13 76	13 71	13 78		
	Fe	13 78	13 72	13 68	13 62	13 58	13 60		
1,000	Al	17 02	17 08	17 00	17 11	17 06	17 00	17 03	
	Cu	17 02	17 01	17 09	17 11	17 07	17 11		
	Fe	17 00	17 03	17 02	16 97	16 94	16 97		
1,500	Al	19 94	19 98	19 91	20 03	19 96	19 92	19 97	
	Cu	19 96	19 92	19 97	20 02	19 99	20 00		
	Fe	19 90	20 02	20 03	20 01	20 02	20 02		
2,000	Al	22 64	22 58	22 62	22 68	22 63	22 62	22 66	
	Cu	22 63	22 60	22 63	22 64	22 66	22 68		
	Fe	22 60	22 68	22 72	22 73	22 74	22 76		
2,500	Al	25 10	25 10	25 10	25 05	25 09	25 08	25 11	
	Cu	25 10	25 09	25 10	25 10	25 12	25 20		
	Fe	25 09	25 08	25 14	25 18	25 16	25 20		
500	Al	14 96	15 12	14 94	15 03	14 96	14 94	14 94	$k_1 = 35$
	Cu	14 96	15 00	15 00	14 97	14 94	15 00		
	Fe	14 97	14 95	14 89	14 85	14 81	14 83		
1,000	Al	17 90	17 97	17 89	17 98	17 93	17 88	17 92	
	Cu	17 90	17 87	17 95	17 98	17 95	18 00		
	Fe	17 90	17 95	17 92	17 88	17 85	17 89		
1,500	Al	20 63	20 60	20 58	20 67	20 60	20 54	20 63	
	Cu	20 63	20 55	20 61	20 66	20 63	20 63		
	Fe	20 58	20 67	20 69	20 69	20 69	20 69		
2,000	Al	23 12	23 06	23 10	23 17	23 12	23 08	23 13	
	Cu	23 12	23 10	23 11	23 12	23 11	23 11		
	Fe	23 06	23 13	23 15	23 19	23 19	23 20		
2,500	Al	25 33	25 44	25 39	25 39	25 41	25 42	25 43	
	Cu	25 40	25 43	25 41	25 39	25 42	25 48		
	Fe	25 41	25.43	25 49	25 51	25 50	25 53		

TABLE 21 —(*Continued*)

$\frac{k_3}{k_2}$	Conductor material	Conductor diameter d_s						Average cable deflection	
		0 999	1 153	1 289	1 412	1 525	1 631		
500	Al	16 23	16 35	16 23	16 35	16 28	16 23	16 25	$k_1 = 40$
	Cu	16 26	16 28	16 31	16 32	16 29	16 32		
	Fe	16 27	16 25	16 23	16 18	16 12	16 17		
1,000	Al	18 92	18 96	18 90	19 00	18 93	18 89	18 95	
	Cu	18 93	18 86	18 96	19 00	18 98	19 01		
	Fe	18 92	18 99	19 00	18 95	18 96	19 00		
1,500	Al	21 39	21 40	21 44	21 48	21 44	21 38	21 44	
	Cu	21 45	21 40	21 40	21 45	21 42	21 43		
	Fe	21 39	21 46	21 50	21 49	21 49	21 51		
2,000	Al	23 77	23 71	23 71	23 78	23 75	23 71	23 77	
	Cu	23 77	23 76	23 77	23 75	23 73	23 78		
	Fe	23 75	23 80	23 84	23 84	23 85	23 86		
2,500	Al	25 91	25 93	25 91	25 88	25 90	25 91	25 95	
	Cu	25 92	25 93	25 90	25 90	25 94	25 99		
	Fe	25 96	25 96	26 02	26 06	26 01	26 04		
500	Al	17 60	17 68	17 58	17 70	17 67	17 57	17 66	$k_1 = 45$
	Cu	17 60	17 60	17 64	17 69	17 63	17 66		
	Fe	17 62	17 66	17 65	17 58	17 55	17 57		
1,000	Al	20 08	20 09	20 05	20 15	20 08	20 02	20 09	
	Cu	20 08	20 10	20 02	20 10	20 14	20 10		
	Fe	20 02	20 10	20 14	20 12	20 10	20 15		
1,500	Al	22 38	22 41	22 33	22 42	22 38	22 36	22 40	
	Cu	22 39	22 35	22 40	22 40	22 38	22 38		
	Fe	22 35	22 42	22 43	22 46	22 43	22 48		
2,000	Al	24 51	24 53	24 56	24 58	24 51	24 52	24 59	
	Cu	24 58	24 58	24 53	24 54	24 56	24 56		
	Fe	24 50	24 53	24 60	24 55	24 65	24 65		
2,500	Al	26 58	26 62	26 55	26 56	26 65	26 64	26 61	
	Cu	26 60	26 62	26 58	26 58	26 60	26 66		
	Fe	26 59	26 58	26 64	26 68	26 61	26 64		
500	Al	19 00	19 06	18 96	19 08	18 96	19 00	19 04	$k_1 = 50$
	Cu	19 02	18 96	19 04	19 10	19 04	19 09		
	Fe	19 02	19 07	19 11	19 05	19 07	19 05		
1,000	Al	21 28	21 29	21 32	21 32	21 27	21 21	21 30	
	Cu	21 30	21 24	21 26	21 26	21 26	21 28		
	Fe	21 24	21 35	21 38	21.37	21 38	21 42		
1,500	Al	23 48	23 38	23 41	23 45	23 43	23 39	23 45	
	Cu	23 43	23 42	23 41	23 40	23 43	23 42		
	Fe	23 42	23 48	23 50	23 52	23 53	23 55		
2,000	Al	25 40	25 48	25 47	25 45	25 48	25 47	25 49	
	Cu	25 48	25 48	25 47	25 45	25 47	25 51		
	Fe	25 48	25 48	25 53	25 56	25 53	25 57		
2,500	Al	27 32	27 41	27 36	27 36	27 42	27 45	27 40	
	Cu	27 33	27 43	27 39	27 38	27 42	27 45		
	Fe	27.42	27 35	27.43	27 46	27 41	27 42		

of maximum economy as found from Eq. (566), reveals the fact that, for any given ratio of $k_3 \div k_2$ and value of k_1, the maximum deflection of the critical catenary is approximately constant for all conductor sizes and all conductor materials lying within the limits covered by the investigation.

In order to substantiate this rather remarkable conclusion, an investigation was carried out covering conductor sizes of from 500,000 to 2,000,000 cir. mils, for each of the three conductor materials mentioned, in which the maximum sags were determined for the most economical spans under critical conditions, and for each of a number of assumed ratios of $k_3 \div k_2$ and k_1. The maximum temperature assumed was 100° F., at which temperature the critical catenary then exists. The results of this investigation are given in Table 21. The data here recorded were obtained by first finding the span of maximum economy in the manner already explained, and then reading from the critical catenary,[1] of the corresponding conductor size and material, the deflection corresponding to this span.

It is thus possible to express the maximum deflection of the critical catenary in terms of the ratio $k_3 \div k_2$ and k_1, regardless of conductor size or material This has been done in the curves of Plate I.

Since the height of tower, to the point of attachment of the conductor, is equal to the required minimum ground clearance plus the maximum deflection on the critical catenary, it follows that the height of tower corresponding to the most economical span, or the *most economical tower height* is given by the equation

$$h_e = k_1 + {}_{\max}y \tag{568}$$

where

h_e = most economical tower height measured from base of tower to point of conductor attachment

k_1 = minimum, allowable, vertical ground clearance.

${}_{\max}y$ = deflection of the critical catenary at center of the most economical span.

To illustrate the significance of the foregoing statements, the most economical spans and the most economical tower heights are computed below for aluminum and copper conductors of random sizes and for given assumed values of $k_3 \div k_2$ and k_1.

Example.—Let $k_3 \div k_2 = 1{,}500$ and $k_1 = 30$. It is required to find the most economical span S, and the most economical tower height h_e, for the following conductors:

1,000,000-cir. mil, aluminum cable for which $d_s = 1.153$ in.
500,000-cir. mil, copper cable for which $d_s = 0.815$ in.

[1] Data for the critical catenaries of all conductors falling within the scope of the table are found in University of Washington Engineering Experiment Station *Bull.* 17.

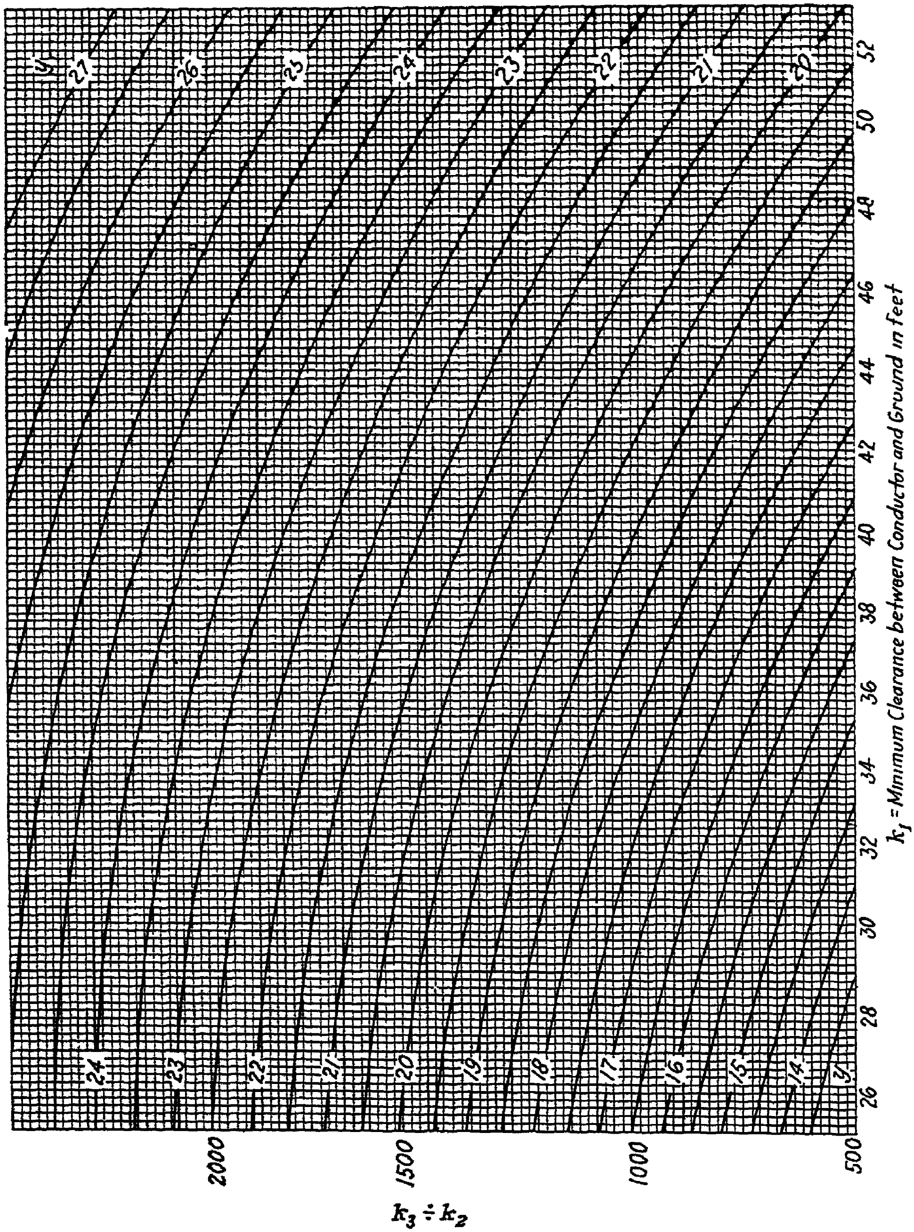

PLATE I.—Curves of maximum conductor sag "y" at center of span for most economical tower spacing.

Solution —From Plates II to V and from Eq. (567), the constants k_6, k_7 and k_8, for the chosen values of $k_3 \div k_2$ and k_1, are

For aluminum conductor

$$k_6 = +0\ 20$$
$$k_7 = 968$$
$$k_8 = 425$$

For copper conductor

$$k_6 = -0\ 15$$
$$k_7 = 832$$
$$k_8 = 112\ 5$$

Substituting these values in Eq (566), for each conductor, yields the most economical spans as follows

1,000,000 cir mils aluminum	$S = 654$ ft
500,000 cir mils copper	$S = 662$ ft

From the critical catenaries for these conductor sizes and materials, and for the spans found, the maximum sags are found to be

$$_{\max} y = 20 \text{ ft for the aluminum cable}$$
$$_{\max} y = 19\ 9 \text{ ft for the copper cable}$$

Turning now to Plate I, it is found that the maximum sag given by the curves for the values $k_3 \div k_2 = 1{,}500$ and $k_1 = 30$, is 20 ft This checks the results of the calculations.

By Eq (568), for the above values of $k_3 \div k_2$ and k_1, the most economical tower height is

$$h_e = 30' + 20' = 50 \text{ ft.}$$

for any size conductor and any conductor material within the range covered by the discussion, that is, for aluminum and copper conductors in sizes between 250,000 cir. mils and 2,000,000 cir mils so long as $k_3 \div k_2 = 1{,}500$ and $k_1 = 30$.

Estimating Value of $k_3 \div k_2$.—It will be recalled that the economic study of this chapter is based on the segregation of tower line costs into seven items as given on page 222 By Eq. (557) the total cost of line supports may be expressed in terms of these items by

$$\text{Cost} = \text{No.}(h^2k_2 + k_3)$$

where

No. = total number of line towers (all towers assumed to be alike).
h = height of tower.
k_2 = (item 1 + item 2 + item 3) $\div h^2$.
k_3 = sum of items 4, 5, 6 and 7.

Thus, in order to estimate what the values of the constants k_2 and k_3 will be, it is necessary to secure, from the manufacturer, cost data on towers designed for a definite conductor tension, voltage and height. From such data the constants for another height, voltage and tension may be estimated if no data are available. The process of making such estimate will be illustrated by an example.

Example.—What are the most economical tower spacing and tower height for a 1,000,000-cir. mil, copper cable strung to a maximum tension of 16,700 lb., insulated for 220 kv. between conductors, and having a minimum ground clearance of $k_1 = 30$ ft ?

It is assumed that, as a basis for making estimates, there are available cost data for a 150-kv. tower line in which the tension of the conductors was 6,000 lb. and the towers were 37 ft. high. These data for the latter are as follows:

1.	Cost of towers at place of erection	\$350 00
2	Cost of erection of towers	70 00
3.	Cost of tower site	15 00
4	Cost of foundation	270 00
5	Cost of location and inspection	25 00
6	Cost of insulators at tower site	80 00
7.	Cost of placing insulators and cable	20 00

Solution.—Using the above data as a basis, the costs for the proposed line are estimated as follows The maximum tension for the proposed line is 16,700 lb, whereas the tension for the line to which the above data apply is only 6,000 lb Since the tower cost is proportional to the tension, the estimated cost of the new towers is

$$\frac{16,700}{6,000} \times \$350.00 = \$975.00, \text{ approximately.}$$

Likewise, the costs of the foundations are proportional to the tensions, whence the estimated cost of the new foundations per tower is

$$\frac{16,700}{6,000} \times \$270.00 = \$750.00, \text{ approximately.}$$

If the number of parallel strings, per conductor attachment, remains unchanged, the cost of insulators is roughly proportional to the voltage, whence the cost of insulators per tower for the new line is

$$\frac{220,000}{150,000} \times 80 = \$117.00, \text{ approximately.}$$

The remaining items of cost remain approximately unchanged. The constants k_1, k_2 and k_3 of Eqs. (556) and (557) may now be found. They are as follows:

k_1 = 30 ft. by conditions of the problem.

k_2 = sum of items 1, 2 and 3 for the new construction, divided by the square of the height of the point of cable support above the tower footing Thus,

$$k_2 = \frac{975 + 70 + 15}{37^2}$$

$$= 0\ 774$$

$$k_3 = \text{sum of items 4, 5, 6 and 7}$$
$$= 750 + 25 + 117 + 20$$
$$= 912$$

and

$$k_3 \div k_2 = \frac{912}{0.774} = 1{,}178.$$

It will be observed that the important items are 1, 4 and 6, and that the final result would not be greatly different if all others were neglected entirely. In the present instance, the use of these three items only would yield

$$k_3 \div k_2 = 1{,}210, \text{ approximately.}$$

With the use of Eq. (566), (567) and (568) and the curves of Plates 2 and 3, the most economical span and tower height readily follow, as in the illustrated example already given. They are

$$S = 683 \text{ ft}$$
$${}_{\max}y = 18\ 1 \text{ ft.}$$
$$h_e = 48.1 \text{ ft}$$

PROBLEMS

1 For a given tower line the ground clearance of the attached conductors is 35 ft. The ratio $k_3 \div k_2$ is 2,000. What is the length of the most economical span for a 350,000-cir. mil, copper cable, whose diameter is d_s = 0.682 inch? What is the most economical tower height for type A tower?

2 The costs items 1, 4 and 6 of a certain 165-kv. tower line. designed for a maximum tension of 8,000 lb. and for a height of 40 ft. to the point of conductor attachment, were as follows:

a. Cost of towers at place of erection............	$420 00
b. Cost of foundations..........................	300 00
c. Cost of insulators at tower site	75 00

Other items of cost may be considered as independent of tension and voltage. Estimate the most economical tower height, the most economical span and the maximum sag for a 110-kv. line having a ground clearance of 30 ft., and which is designed for a maximum tension of 6,000 lb. on towers.

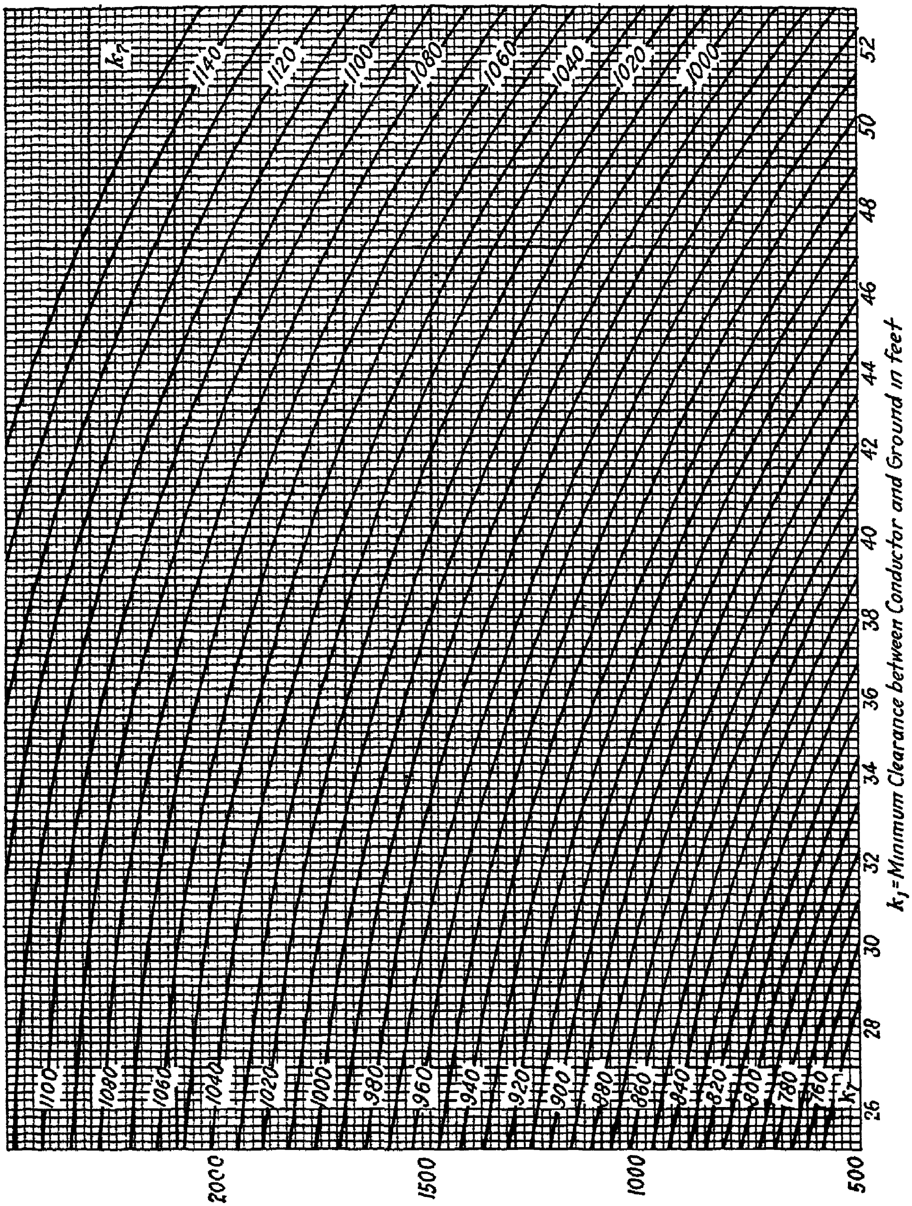

PLATE II.—Curves of constants k_7 for aluminum conductors.

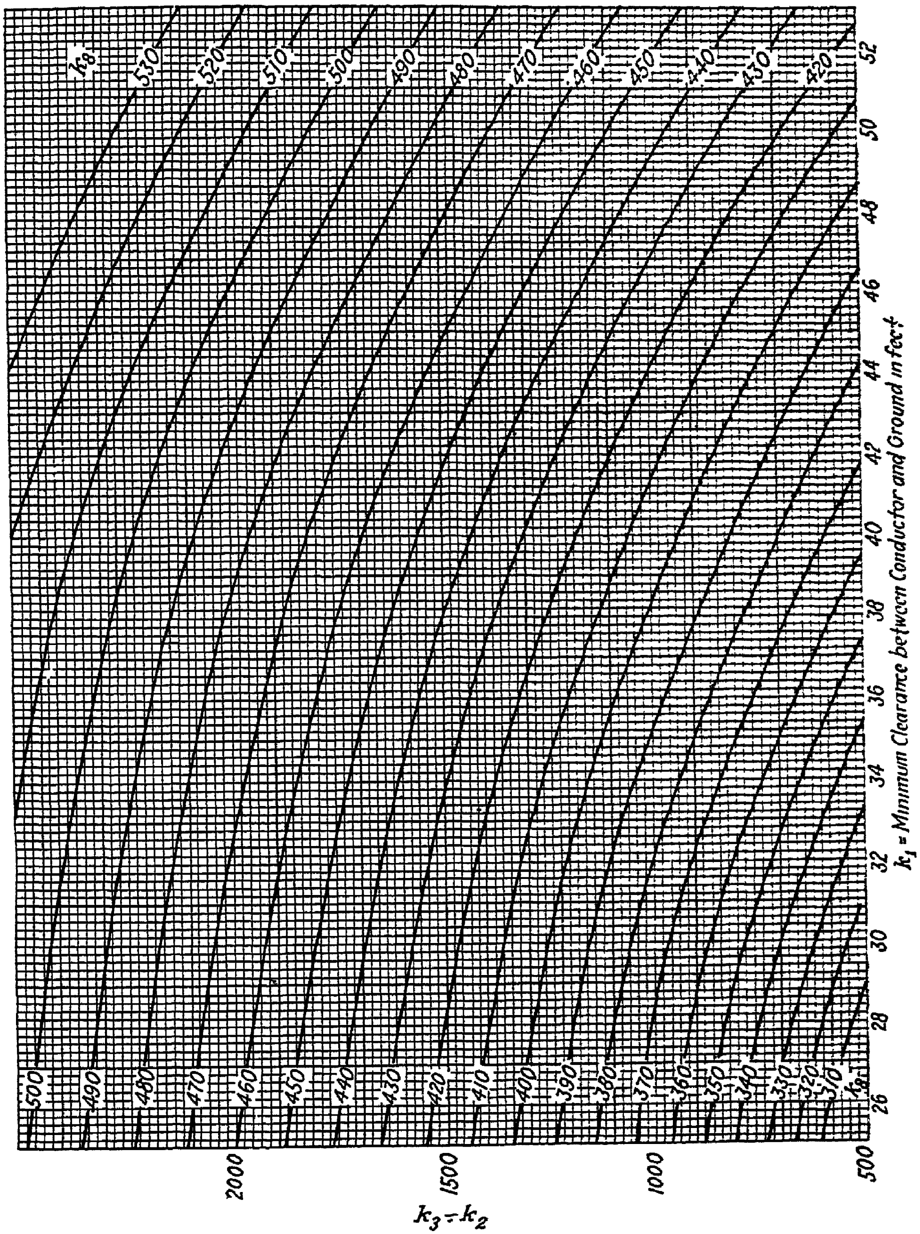

PLATE III —Curves of constants k_8 for aluminum conductors.

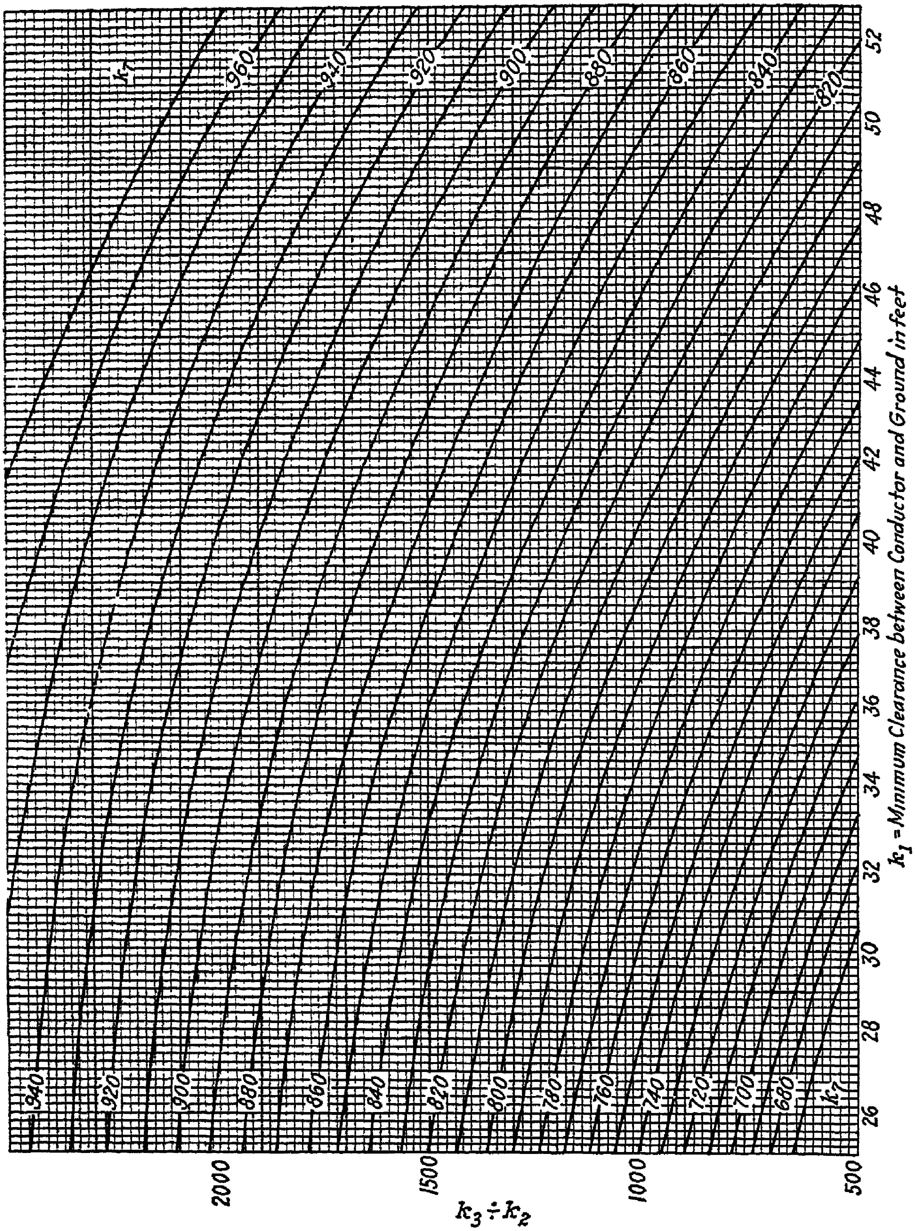

PLATE IV.—Curves of constants k_7 for copper conductors.

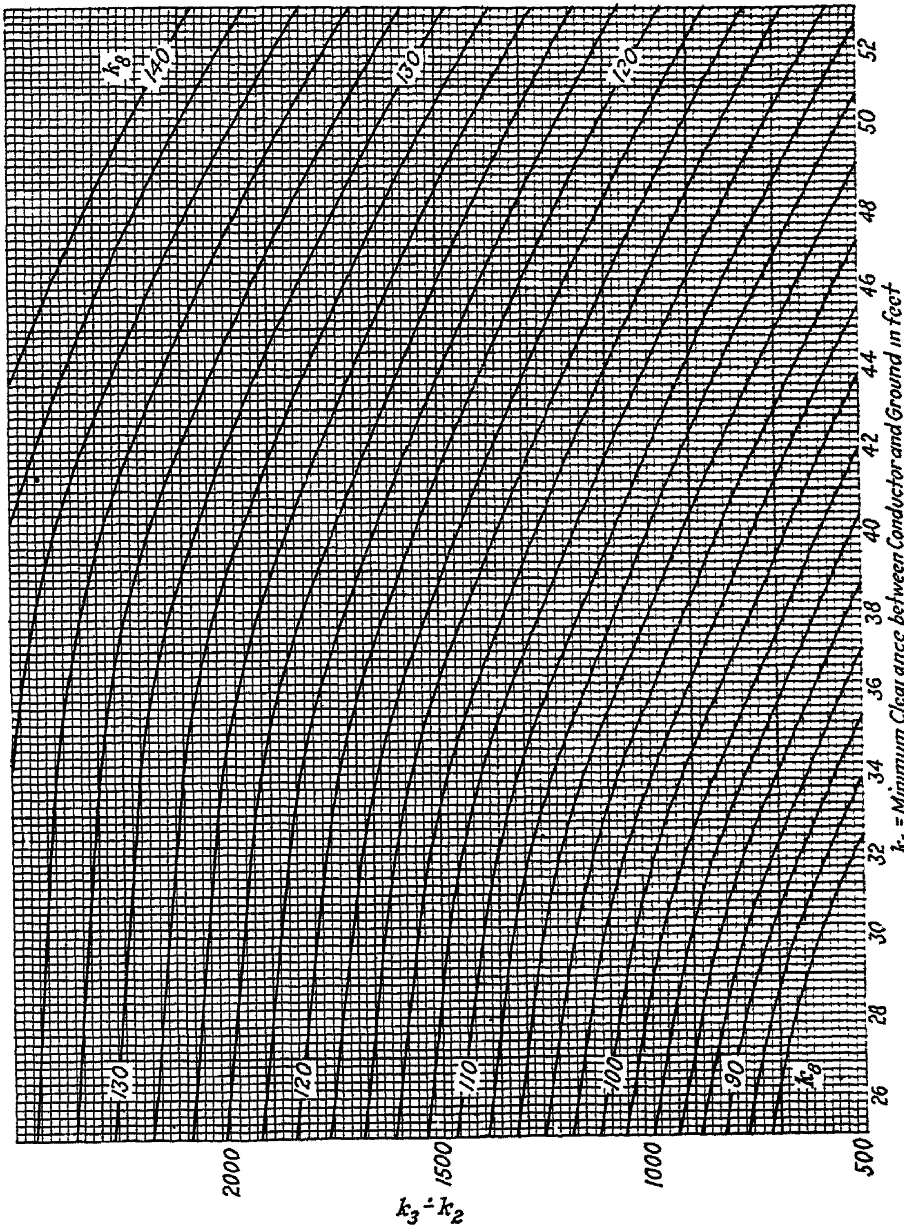

PLATE V.—Curves of constants k_8 for copper conductors.

CHAPTER XIII

THE MOST ECONOMICAL VOLTAGE AND CONDUCTOR DIAMETER[1]

Given the location of the project, the load to be transmitted and the distance of transmission, the problem of designing a long line divides itself naturally into three parts; namely

1. Determination of the most economical voltage and conductor size.
2. Selection of the most economical tower designs and spans, together with the proper design of span from a mechanical standpoint.
3. Choice of synchronous reactors for line regulation, and the calculation of the electrical performance of the line.

The second of these two items has already received some attention in Chaps. XI and XII. Much of this will serve as a prerequisite to the present chapter. The first item, namely, the formulation of a method by which the most economical voltage and conductor diameter may be found, is the burden of the present chapter.

While the design of important transmission lines is usually placed in the hands of experts of wide experience whose judgment may frequently come to their assistance where rational methods fail or do not exist, there is always danger in relying too much upon judgment which cannot be verified by careful engineering calculations. And, strangely enough, the very part of the problem of calculating the design of a transmission line, where the careful application of the underlying scientific principles is of first importance, is the place where greatest uncertainty usually exists, namely, in the calculation of the most economical voltage and conductor areas for a given project.

Many have contributed towards systematizing and rationalizing the general procedure in transmission-line design, and much

[1] "Transmission Line Design II. The Line of Maximum Economy," A. I. E. E., *Jour.*, 1925, KIRSTEN, F. K., and E. A. LOEW; also University of Washington Engineering Experiment Station *Bull.* 32.

has been accomplished in the way of supplying solutions to the various problems encountered, particularly in reference to items 2 and 3 above. While the procedure with reference to these items is quite clearly established and well defined, the same is not true of item 1. The solution here is somewhat more involved, and, while the underlying laws are simple enough, their proper application is quite difficult. Here the usual procedure has followed the method of trial and error, and in the final analysis, much was left to the uncertainty of the engineer's judgment. The object of the present chapter is to rationalize the entire procedure as affecting item 1, so far as is possible, and to provide a method of solution that is based on scientific laws, and in which possible errors, such as are likely to result from placing too great a tax upon the designer's judgment, are reduced to a minimum.

In the past it has been the common practice to treat the electrical and the mechanical features of the line separately as though they were quite unrelated quantities The size and the material of the conductor were first determined, often by a false application of Kelvin's law, in which no consideration of the most suitable line voltage, of the cost of high-tension equipment, or of tower structures entered. Following this, the proper number of towers, their height and weight necessary to support the conductors, with the required ground clearances, were calculated. Usually the operating voltage of the line was more or less arbitrarily fixed, and the choice of conductor material gave evidence not so much of an intimate knowledge of the economic laws involved as of the relative efficiencies of the sales departments of the conductor manufacturers. This procedure has resulted in the construction of lines which do not give maximum service at minimum cost, although Kelvin's law has apparently been applied in making the design computations.

This chapter undertakes to give the engineer a systematic method of attack upon the problem of finding the most economical voltage and conductor diameter, so that he may be assured of supplying the required service at minimum cost.

Kelvin's Law and Its Modification.—The determination of the most economical size of conductor is based on the well-known principal stated in Kelvin's law. The statement of this law has been somewhat modified since it was first proposed, but, even in its modified form, it is now sometimes improperly inter-

preted. The law was first stated by Sir William Thompson (Lord Kelvin), in 1881, but was later modified by Gisbert Kapp into the following more exact form: "The most economical area of conductor is that for which the annual cost of energy wasted is equal to the interest on that portion of the capital outlay which can be considered proportional to the weight of copper (conductor) used." The term interest used in the above statement of the law should be interpreted to mean interest and depreciation.

The law as stated assumes the line voltage to be fixed; that is, the relative economy of conductors of different sizes is to be compared on the basis of an assumed voltage. When a problem in power transmission is first attacked, however, the voltage is usually as much an unknown as is the size of the conductor to be used. A proper choice of conductor size must accordingly be intimately interlinked with a choice of the most suitable transmission voltage. *The problem, therefore, is to find that conductor which, when used with the most economical voltage, will fulfill the requirements of Kelvin's law.*

The truth of this statement is supported by the following argument. If a fixed loss in the line conductors be assumed, the cross-sectional area of conductor required to transmit a given amount of power is inversely proportional to the square of the transmission voltage. As the voltage is increased, the investment in line conductor is reduced. One cannot go on increasing the voltage and reducing the size of conductor indefinitely, however, because a definite limit is set by two controlling economic conditions. The first limit is set by the fact that for every voltage and conductor spacing, other factors being assumed constant, there is a definite size of conductor which is the minimum that may be used. Any further reduction in size will give rise to corona loss from the conductors and will tend to cause incipient sparking or flashovers, thus endangering the line insulation. A second limitation resides in the fact that, as the voltage is increased, the total investment, in the equipment, required to handle a given amount of power increases very rapidly. This is particularly true of transformers, switches, lightning arresters and insulators. *When that point is reached, beyond which any further increase in voltage will entail an increase in annual fixed charges, on high-tension and line equipment, greater than the corresponding decrease resulting from a reduction in size of line conductor,*

there is no further advantage in increasing the voltage. At the point thus determined, the most economical voltage and the most economical conductor size are both found.

Items of Cost Affecting Choice of Voltage and Conductor.—The final choice of conductor will not necessarily be based on the above considerations alone. Other conditions which may affect the size of conductor are regulation, charging current and the cost of auto-transformers required for linking the line with established systems operating at different voltages. Where synchronous reactors are installed to take care of line regulation, due consideration must also be given to the interdependence of the operating voltage and synchronous reactor capacity required for a desired regulation.

Hence the question, as to what voltage should be used on a transmission system carrying a given amount of energy, can only be answered correctly after investigating all cost items of the line and of its operation which are a function of the voltage, as intimated above. Some of these items are: Heat loss, insulators, transformers, high-tension oil switches and disconnecting switches, lightning arresters, housing space for high-tension apparatus, synchronous condenser capacity, tower dimensions, etc.

In the following analysis the above items are grouped under six main headings, namely

1. Load distribution.
2. Line conductors.
3. Conductor supports and line insulators.
4. High tension apparatus and housing.
5. Regulation.
6. Coordination of existing systems to new project by tie lines.

Load Distribution.—Since the conditions, pertaining to the generating station or stations feeding a transmission line, as well as to the load served, are so variable as to make each line a separate and distinct problem, differing in some features at least, from every other line, no general rules can be laid down for estimating the amount of power to be transmitted over the line, which will apply with equal force to all lines. Each project must be studied separately in the light of all the facts available, and from these an estimate may be reached of the service which the lines should render. In certain sections of the country, where waterpower is abundant, transmission of power is usually

associated with its generation in hydroelectric plants. In fact, a considerable percentage of our principal power transmission projects are fed largely from plants of this kind. A method of estimating the load of a proposed line, based on plants of this kind, will therefore serve to illustrate the procedure for a large number of lines, and, perhaps, at the same time, suggest a suitable approach to the problem of finding the required duty of most others.

a. Rated Ultimate Capacity of Generating Machinery.—The basic data, for the designer of a waterpower project, are the total energy obtainable per year from the generating machinery at the power site, together with the curves indicating its probable distribution from week to week throughout the year. These data are obtained from a careful study of the stream-flow measurements taken over a period of years, and a study of the water storage possibilities for the proposed development The method of analyzing these data need not be considered here. With a given maximum elevation of the storage dam, and with a knowledge of the conditions of flow into, and the amount of storage available in the storage reservoir, a fairly close estimate of the ultimate output of the generating station can be made.

The capacity of the generating machinery, however, depends not only upon the average amount of energy obtained by averaging the total available yearly energy over the period of one year, but it also depends upon the distribution of the daily demand at the receiving end of the transmission line over the hours of the day. Since the load demand of the distribution system usually fluctuates over a wide range during the day, and since this variation differs from day to day and from month to month, the generating machinery must have enough capacity to supply the maximum annual peak instead of the average daily demand Where auxiliary plants are provided to supply the peak demand of the distribution system, the required capacity of the generators at the power house is more nearly equal to the average yearly demand.

b. Load Factor.—The ratio, of the average daily power demand for the year to the maximum or half-hour peak demand upon the generators for the year, is called the *average half-hour, yearly load factor* of the power station. For western coast cities this factor usually lies between 40 and 65 per cent, depending upon the nature of the load connected to the distribution system.

c. Distribution Curve.—The load factor can only be approximated by the designer after a careful study of the probable, ultimate, daily load distribution from past records of the distribution system. From this study the total ultimate capacity, in kilovolt-amperes of the generating station is obtained by dividing the average daily energy available by 24 times the load factor. If the maximum daily peak demand is of short duration, the generating machinery may be operated under an overload for that period, thereby increasing the load factor.

Having determined the average daily load distribution of the existing distribution system, the average, daily distribution curve for the ultimate system, including the new development, may be found by multiplying the ordinates of the original curve (kilowatts) by a constant, of such magnitude that the area under the new curve, in kilowatt-hours, represents, for the ultimate system, the average daily supply from all power sources feeding the system in question The constant is quickly determined by integrating the given load-distribution curve with the planimeter and dividing the average daily energy yield of the generating system by this integrated area in kilowatt-hours. The capacity of the generating machinery is found by multiplying the maximum ordinate of the derived curve by the ratio of the maximum peak of the year to the maximum peak of the average day. Other considerations, as auxiliary plants, peak load, power delivered over tie lines from other systems, concurrence of high water conditions with maximum peak load demands, etc , may modify the determination of the generating capacity considerably and must be studied separately for each individual project.

Root-mean-square Kilowatts and Average Heat Loss in the Line.—Having derived the average daily distribution curve for the year, it is now possible to determine the "root mean square" of the average load. The root mean square of the average load is that quantity which, if supplied continuously throughout the year, would give rise to the same total heat loss in the line conductors as is dissipated in them under actual operating conditions. Since the heat energy dissipated in the line conductors is proportional to the square of the line current, assuming constant voltage and power factor, the line loss at any moment is proportional to the square of the power transmitted, and the average line loss for the year is proportional to the average square of the power transmitted, hour by hour throughout the year. The

r.m.s. average load is then found by taking the square root of the average square of the ordinates to the mean daily load curve for the year.

Figure 74 shows in full line a typical mean daily load curve for a western community. The ordinates can be drawn to any arbitrary scale representing kilowatts. From this curve

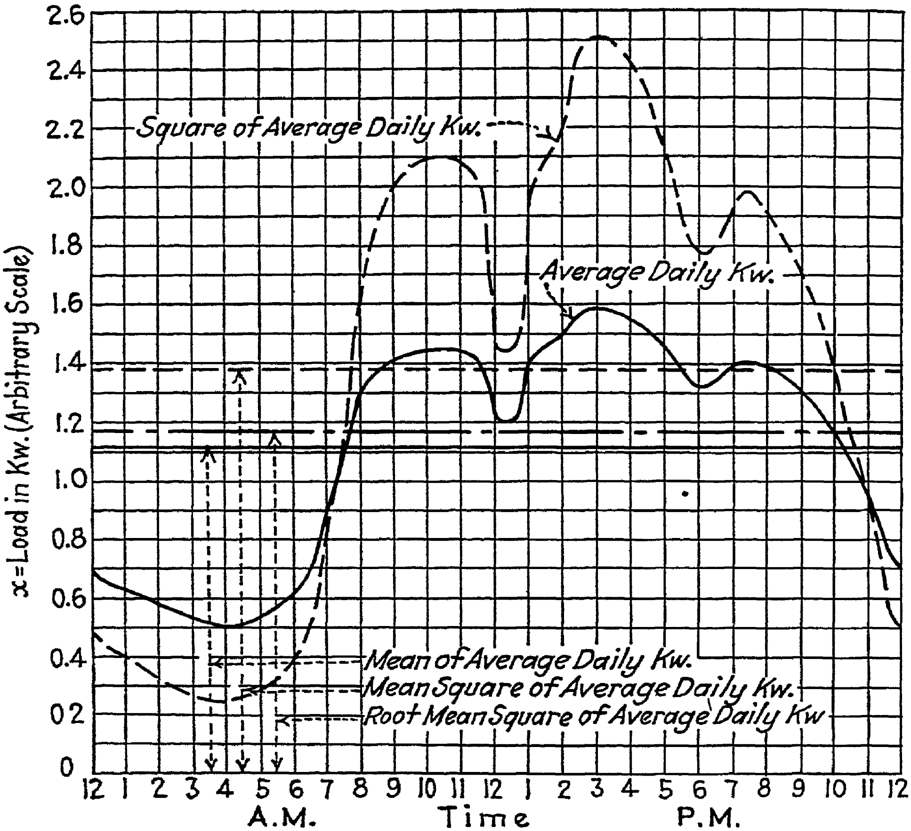

FIG. 74.—R.m s. kilowatts from average daily load curve.

the (kw.)² curve, shown dotted, is derived by squaring the ordinates of the kilowatt curve. Applying the planimeter to the latter curve and dividing the integrated area by 24 gives the average (kw.)² for the day. As a check, the area below the horizontal dotted line should equal the area of the (kw.)² curve. The square root of the average (kw.)² is that amount of energy which, if transmitted continuously, will give rise to the same heat loss as the variable load actually transmitted under operating

conditions. The quantity thus derived will hereafter be called the *root-mean-square kilowatts.* (569)

Figure 74 shows that the r m s. kilowatts is a quantity greater than the average daily kilowatts

Line Conductors.—Under this heading should be collected all data pertaining to the conductors, which may be of value in finding the cost of the energy annually wasted as line loss, the annual charge against the first cost of conductor, and the influence of the mechanical features of the conductor upon the cost of the line towers. Since there is usually a choice of several conductor materials, as for instance, copper, aluminum and steel and also a choice of composite cables of steel-core aluminum, copper-clad steel, or of cables of special mechanical construction, such as hollow cables made up of a spiraled inner tube with cable strands spiraled on the surface of the tube, the electrical and mechanical characteristics of all cables which come into consideration should be obtained.

The electrical characteristics of a given cable at once fix the line loss per kilowatt-hour per unit metallic section of the cable. The line loss is determined as follows:

a. Energy Loss.—The total power wasted in the conductors of a three-phase line is

$$P = 3RI^2$$

where

P = watts loss in the line.
R = the total resistance of a single line conductor in ohms.
I = equivalent average current in amperes which will give rise to the same heat loss per year as the integrated losses due to the actual current of the daily variable load

The energy wasted during the year is

$$\begin{aligned} W &= Pt \\ &= 3 \times 8{,}760RI^2 \text{ watt-hrs.} \end{aligned} \tag{570}$$

and the value of the annually wasted energy is

$$\begin{aligned} \text{Cost} &= \frac{3 \times 8{,}760RI^2A}{1{,}000} \\ &= 26.28RI^2A \text{ dollars} \end{aligned} \tag{571}$$

where

A = value of electrical energy at the receiving end of the line, in dollars per kilowatt hour.

If a stranded conductor be used, remembering that its diameter[1] $d_s = 1.153d$, where d_s and d are the diameters of stranded conductor and equivalent solid rod, respectively, in inches,

$$R = \frac{1.02\rho L(1.153)^2 \times 10^{-6}}{d_s^2} \tag{572}$$

where

ρ = resistance of conductor material in ohms per mil-foot

L = actual length, in feet, of the line conductor measured from the generating station to the receiving station along the catenaries formed in suspension.

1.02 = A.I.E.E. resistivity factor introduced to take care of increased resistance due to stranding.

Substituting Eq. (572) into (571),

$$\text{Cost} = 35.64 \times 10^{-6}\rho LA\frac{I^2}{d_s^2} \text{ dollars.} \tag{573}$$

Equation (573) gives the value of the energy lost in transmission in terms of two variables for a given conductor material. These two variables are the equivalent average current I, and the overall diameter of the cable in inches. If composite cables are used, the ratio, of the area of a circle drawn tangent to the outermost elements of the surface layer of strands, to actual metallic section, is $(1.153)^2$ approximately, provided the stranding of these cables is the same as that of cables made of one kind of material. The constant ρ, however, must be the derived electrical resistance per mil-foot of some equivalent material which would yield the same resistance if made up into a cable of the same diameter as the composition cable. If steel is a part of the conductor, the resistance is a function of the frequency. Special attention should here be called to the fact that the constant 1.153 can only be used in connection with cables of standard stranding. The constant, for cables of special mechanical construction, such as hemp-core, tubular conductors and cables of special stranding, must be derived and is the ratio of the diameter of a circle enveloping the extreme outer elements of the cable to the diameter of the equivalent solid rod. It is important to state Eq. (572) in terms

[1] The ratio $d_s \div d$ varies between 1.1508 and 1.536 for the larger cables as shown on p. 61. The value 1.153 is taken as the correct value for the larger cables.

of the overall diameter of the cable, as will be apparent from subsequent derivations.

b. Fixed Charges against Line Conductors.—The volume of the three cylindrical line conductors is

$$V = \frac{3\pi d_s^2 L}{4 \times 144 \times (1.153)^2} \text{ cu. ft.} \tag{574}$$

and the annual fixed charge against the conductor alone is

$$\text{Conductor fixed charge} = \frac{VBW}{100} \text{ dollars} \tag{575}$$

where

p_1 = per cent interest and depreciation

B = cost of conductor material delivered, in dollars per pound.

W = weight of conductor material in pounds per cubic foot.

Substituting Eq. (574) in (575),

$$\begin{aligned}\text{Conductor fixed charge} &= \frac{3\pi d_s^2 p_1 BWL}{4 \times 144 \times (1.153)^2 \times 100} \text{ dollars} \\ &= 12.31 \times 10^{-5} d_s^2 p_1 BWL \text{ dollars}\end{aligned} \tag{576}$$

Equation (576) shows that the annual fixed charge against the conductor alone, for a given conductor material, is a function of the square of the overall diameter d_s, which appears as the only variable

Conductor Supports and Line Insulators. (*a*) *Types of Towers.* For long-distance, high-potential lines, only two general types of towers need be considered by the designer, namely the single-circuit and the double-circuit tower. Figure 75 shows the two types Type *A* represents the single-circuit tower with the three conductors arranged horizontally on one crossarm, and type *B* the double-circuit tower with the two circuits arranged vertically, one on each side of the tower, supported by three crossarms. On the west coast of America, where timber is still plentiful, the wood-pole tower is still an economic temptation, but with the gradual increase in timber cost and an increasing demand for greater permanency of construction for important trunk lines, this type will soon vanish from the field of long-distance transmission, and, for that reason, is not considered here.

(*b*) *Comparison of Towers Type A and Type B. Mechanical Features.*—For a given conductor clearance to ground, type *B*

tower is much higher than type A. Therefore, the wind load on the conductors and the structure is much greater than for type A tower. This necessitates an increased weight of all members and of the foundation. Furthermore, tower B must be designed for the tension of six conductors, and this tension being very nearly twice that for which tower A is designed, acts, in addition, with a much greater moment about the foundation. In case of

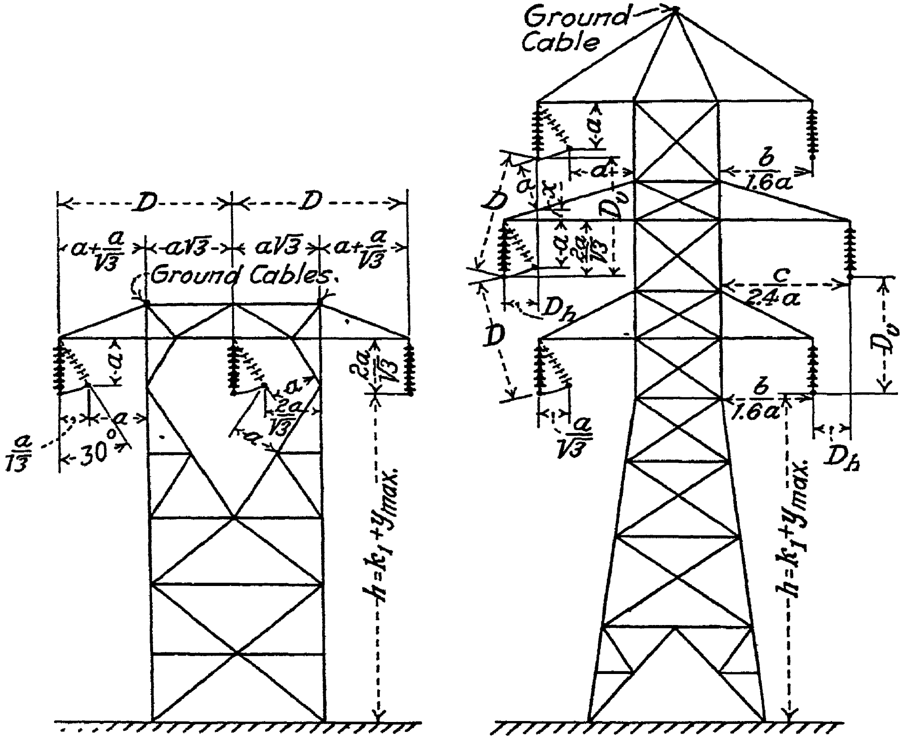

FIG. 75.—Single circuit and double circuit towers.

mechanical failure of one circuit on the double-circuit tower, a possibility which must be considered, the total tension of the remaining three conductors acts as a twisting moment on the tower structure, whereas the total twisting moment for type A tower is that due to the tension on one conductor only. Hence, for the same conductor, the structural members of the double-circuit tower must be heavier than those of the single-circuit tower. The width of the line right-of-way is materially decreased

by using tower type *B*, however, and this item may prove a most important factor in the choice between the two types. Another factor in favor of tower *B* is the need of only one ground cable which, when placed at the apex of the upper crossarm, furnishes protection against lightning disturbances for two circuits, whereas type *A* tower requires two ground cables located on the upper crossarm, as shown in sketch, so that the number of ground wires per line is four times as great for type *A* as for type *B*. Very often only one cable is used for type *A*, but two cables are more desirable from the standpoint of reducing twisting moments on the tower structure and foundations, and distributing these moments over adjacent towers in case of conductor failures. The distribution of such stresses to adjacent towers, by the ground cable of type *B* tower, is less effective in relieving the tower structure, especially in case of failure of the upper conductors, in which case the stresses must pass through half the tower structure before they can be transmitted to other towers by the ground cable.

c. Comparison of Type A and Type B Towers. Electrical Features.—It will be shown later that, for a given line voltage, the spacing between conductors is less for type *B* than for type *A* towers. Hence the inductance per phase is less and the capacitance greater than for type *A*. This may prove an advantage or disadvantage, depending upon the length of line and the character of its electrical load. For short lines of large capacity it is an advantage and for long lines of relatively small capacity it is likely to be a disadvantage.

Due to the offset from a vertical plane of the center conductor, the tower *B* arrangement approaches equilateral conductor arrangement, and hence, the phase performance along the line will be more nearly balanced for the three conductors than for the conductor arrangement of tower *A*. The phase unbalance is further aggravated by tower *A*, inasmuch as the center conductor must pass through a closed iron circuit, giving rise to hysteresis and eddy-current loss, which, however slight, is entirely absent in tower *B*.

Danger of insulator flashovers becoming short circuits between conductors is very improbable for type *A* towers, whereas these flashovers have a natural tendency to spread vertically, thus tending to aggravate the destructive effects of these flashovers for type *B* towers.

For very high potential lines on type *B* towers, repair work on one line may necessitate the shutting down of the other line also, to insure safety to the workmen. This would make the use of type *A* towers not only desirable but almost necessary.

Type *A* tower lends itself very much better than type *B* to a step development of large projects if the magnitude of each step approximates the normal capacity of one line only.

It is not desirable to use different types of transmission towers for parallel lines conveying the energy of a power source to the same distribution center, since, as explained above, the electrical characteristics of these lines would differ.

From the above it is evident that the designer can arrive at the proper choice of tower type only after careful consideration of all factors involved, especially factors of cost; and the cost factors should be analyzed for both type *A* and type *B* towers before giving too much weight to the saving in right-of-way or to the desirability of least possible service interruption The choice of tower type also influences the size of conductor required to carry a given amount of power, as will be demonstrated in the subsequent discussion.

d. Conductor Spacings—The clearances which must be provided between the conductor of a line, or between line and ground, are determined largely by considerations of electrical safety. Sufficient clearance should be provided to allow a reasonable factor of safety against sparkover between conductors, or between any one conductor and the tower structure, under the worst weather conditions and with one conductor grounded. In regions where large birds may be the cause of electrical failure the separation of conductors should be sufficient to prevent the likelihood of short circuits from the wings of such birds.

Where two circuits are strung on a single line of towers, the middle conductor is usually offset from the vertical plane containing the upper and lower conductors. The offset is usually away from the tower and amounts to from 2 to 4 ft. This is done to prevent short circuits in a given span due to temporarily decreased clearances which might arise from the dropping of the sleet load from one or more conductors. The swinging of conductors due to the wind is not usually regarded as a serious problem particularly with heavy conductors, since they swing together and maintain more or less constant clearances. The vertical clearances between conductors are usually somewhat less than

the horizontal. The spacing of conductors depends also somewhat upon the length of the span. The clearance between conductor and tower structure must take account of the possibility of the swing of the conductor with the wind. The probable angle of swing is estimated, and the clearances are chosen accordingly While there is considerable divergence in the practice followed by different companies, the curves of Fig. 76 probably represent present-day average practice fairly well.

e Clearances to Ground—Clearances of conductors to ground vary considerably with the location and with the necessity for taking precautions. They should always be sufficient to avoid

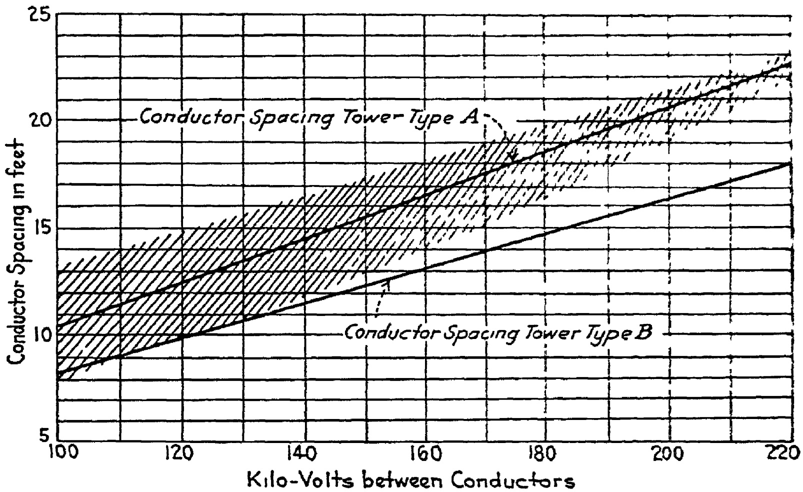

FIG. 76.—Conductor spacings

the possibility of harm coming to persons, or damage to property from the proximity of other electrical lines. Over highways, railways or waterways special precautions and greater clearances than ordinary are required In open country from 20 to 25 ft. is usually deemed sufficient. In some states clearances are specified by law. Recommended ground clearances as well as conductor spacings may be found in the National Electrical Safety Code.

Factor of Safety.—The actual sparkover distance between conductors is approximately 1 in. per 10,000 volts. Most systems go so far as to provide for safe operation with one conductor grounded. For that emergency, the voltage between the remain-

ing conductors and the tower structure is not the voltage to neutral, but the full voltage normally between conductors The grounding to the tower of one of the line conductors naturally decreases the electrical safety factor of the line insulation in the proportion of $1 \div \sqrt{3}$, so that, if the normal safety factor of the line be 6.5, it would be reduced to $6.5 \div \sqrt{3}$ or to 3.75 in case of the grounding of one conductor. Using a factor of safety of 6.5, for long lines under normal operation, will provide 6.5 in. of clearance for every 10,000 volts of operating potential This will assure sufficient safety for transient potential waves of considerable magnitude.

The choice of a safety factor of 6.5 is rather arbitrary, but seems to fit present practice very well, as shown by Fig 76. The shaded area represents the extreme ranges of general spacing practice, as found from many present installations, and represents the maximum range as reported in the handbook of the Aluminum Company of America, on page 72. The conductor spacing curves for type A and type B towers were obtained by applying the safety factor 6.5 to both types of towers in connection with actual sparkover distances under extreme wind deflections of the insulators.

f. Empirical Equation for Spacing D.—Referring again to Fig. 75, the minimum clearance of the conductor to the nearest tower member under maximum wind load on the cable is designated a. With the assumption that the maximum deflection of an insulator string is 30° from its normal vertical position, and that, for type A tower, the structural members which form the tower frame on each side of the central conductor are also inclined at an angle of 30° from the vertical, the spacing between adjacent conductors is

$$
\begin{aligned}
D &= a + \frac{a}{\sqrt{3}} + a\sqrt{3} \\
&= 3.31a \quad \text{for type } A \text{ tower.} \qquad (577)
\end{aligned}
$$

The diagram of type B tower shows the center conductor projected out a distance designated as D_h from the vertical plane which intersects the other two conductors. This is necessary in order to provide safe clearance near the middle of the cable span in case of unequal sleet loading of the three conductors.

For the type B tower it is assumed that the ratio of the projection upon a vertical plane of the actual conductor spacing D,

designated D_v, to the projection of D upon a horizontal plane, designated D_h, is fixed as $3 \div 1$. Likewise, it is assumed that the slope of the upper member of the middle crossarm with respect to a horizontal plane is $1 \div 3$. From these assumptions, the spacing between adjacent conductors is

$$D_v = \frac{3D}{\sqrt{10}}$$

$$= \frac{2a}{\sqrt{3}} + x\frac{a\sqrt{10}}{3}.$$

But

$$x = \frac{D_h}{3}$$

$$= \frac{D}{3\sqrt{10}}$$

hence

$$D = 2.62a \text{ for type } B \text{ tower.} \tag{578}$$

Designating the operating voltage to neutral of the line by E_n, and allowing $10{,}000 \div 6.5$ volts per inch,

$$a = \frac{E_n}{1{,}538} \text{ in.} \tag{579}$$

Combining Eq. (579) with Eqs. (577) and (578),

$$\left.\begin{aligned} D &= 0.00215E_n \text{ for type } A \text{ tower} \\ D &= 0.00170E_n \text{ for type } B \text{ tower} \end{aligned}\right\} \tag{580}$$

Any other special arrangement of conductors, on towers different from the general forms, shown by Fig. 75, may be similarly analyzed for a relationship of E and D, resulting, probably, in a change in magnitude of the constants of Eq. (580).

g. Equivalent Spacing.—To equalize the electrical performances of all three phases of a transmission circuit, the conductors of which are unequally spaced, as exemplified by Fig. 75, it is necessary to provide a minimum of two complete transpositions, the transposition points dividing the line into sections of approximately equal length. Additional transpositions may be desirable or even required by law, in order to reduce to a safe working value inductive interference with adjacent telephone circuits. A circuit of unequal conductor spacings, thus transposed, will

perform like a three-phase circuit without transpositions, the conductors of which are equidistant from each other, *i e.*, arranged so that lines through their axes form an equilateral triangle, provided that

$$D' = \sqrt[3]{D_1 \times D_2 \times D_3} \tag{581}$$

where D_1, D_2 and D_3 are the actual distances, center to center, of the three line conductors properly transposed, and where D' is the equivalent equilateral spacing of the same circuit without transpositions but yielding the same performance. The proof of this relationship is found in Eqs. (160) and (252).

The equivalent triangular spacing, as applied to Fig 75 is, therefore,

$$\left.\begin{aligned} D' &= \sqrt[3]{2D^3} \\ &= 1\ 260D \text{ for type } A \text{ tower} \\ \text{and} \qquad & \\ D' &= \sqrt{\frac{6D^3}{\sqrt{10}}} \\ &= 1\ 238D \text{ for type } B \text{ tower} \end{aligned}\right\} \tag{582}$$

h. Relation of Transmission Voltage and Conductor Diameter.—It has already been stated that in a proper application of Kelvin's law all items of cost, which enter into the completed project and whose values depend upon either the line voltage or the conductor area, must be considered. For convenient use in the solution of problems it is most desirable to express the law in the form of a mathematical equation in which the conductor diameter appears as an explicit function of the load to be transmitted. To do this requires the formulation of the mathematical law by which the voltage and conductor diameter, in an economically designed line, are or should be related. This is so because certain of the cost items in question, such as the cost of transformers, circuit-breakers, towers, insulators, and lightning arresters, are usually supplied by the manufacturer for a given operating voltage. The law, of variation of cost with voltage for these items, is built up from the manufacturers' quotations, as will be explained in greater detail later. The law of variation of cost may then also be expressed as a function of the conductor diameter as soon as the relation which should exist between conductor diameter and line voltage is known.

It was explained at the outset that, if the conductor diameter is assumed to be fixed, the line losses incident to the transmission of a given load over the line are a minimum, when the voltage impressed is the highest practical value, *i.e.*, when the voltage is just under the critical disruptive value. Therefore, the basic assumption is made that *for maximum economy all lines should be operated at an average voltage somewhat below the critical, disruptive value for the conductor used,* but yet as high as is practical.

The interrelation of voltage and conductor diameter sought is expressed by the law of the corona (Eq. (287)) as follows:

$$_cE_n = 2.302 m_0 g_0 \delta \frac{d_s}{2} \log_{10} \frac{2D'}{d_s}$$

where

$_cE_n$ = the critical disruptive voltage to neutral in volts.
m_0 = the irregularity factor.
g_0 = the dielectric strength of air at standard temperature and pressure in volts per inch.
= 53,600 volts (effective value) per inch.
δ = the altitude factor.
d_s = the diameter of stranded conductor in inches
D' = the equivalent equilateral triangular spacing between line conductors in inches.

If the ratio, of critical disruptive voltage to actual line voltage, be represented by γ, the relation between the two voltages may be expressed by the equation

$$E_n = \gamma_c E_n. \tag{583}$$

For short lines, where control of the generator excitation alone is largely depended upon to maintain constant voltage at the receiving end, γ may have to be as low as 0 8 or 0.85 in order to maintain a suitable difference between actual line voltage and corona voltage at all loads. For longer lines or, in general, lines in which phase control is employed, γ may well be 0 85 to 0.90.

Substituting E_n from Eq. (583), the law of the corona becomes

$$E_n = 1.151 \gamma m_0 g_0 \delta d_s \log_{10} \frac{2D'}{d_s}. \tag{584}$$

For all practical long-distance transmission circuits the quantity $2D' \div d_s$ lies well within the limits of 100 to 1,000 and, consequently, the logarithm of this quantity is never less than 2 nor greater than 3. Within this limited range, a graph, expressing

the relationship of the logarithm of $2D' \div d_s$ to the quantity $2D' \div d_s$ itself, is almost identical with a parabola of the form,

$$y = 2 + 0.034\left(\frac{2D'}{d_s} - 100\right)^{\frac{1}{3}}. \tag{585}$$

This statement may readily be verified by calculation as shown in Table 22.

TABLE 22

$\frac{2D'}{d_s}$	$\log_{10} \frac{2D'}{d_s}$	$2 + 0.034\left(\frac{2D'}{d_s} - 100\right)^{\frac{1}{3}}$
100	2 00	2 03
200	2 30	2 34
300	2 48	2 48
400	2 60	2 59
500	2 70	2 68
600	2 78	2 76
700	2 84	2 83
800	2 90	2 91
900	2 95	2 96
1,000	3 00	3 02

The above table shows that the value y computed from Eq. (585) may be substituted for the logarithm of $2D' \div d_s$ for all long transmission lines.

Substituting y from Eq. (585) into Eq. (584) for the value of $\log_{10} \frac{2D'}{d_s}$,

$$1.151\gamma m_0 g_0 \delta d_s\left[2 + 0.034\left(\frac{2D'}{d_s} - 100\right)^{\frac{1}{3}}\right] \tag{586}$$

and, incorporating the relationships expressed by Eqs. (580) and (582), Eq. (586) becomes

$$E_n = 1.151\gamma m_0 g_0 \delta d_s\left[2 + 0.034\left(\frac{k_9 E_n}{d_s} - 100\right)^{\frac{1}{3}}\right] \tag{587}$$

where

$$k_9 = 0.0054 \text{ for type } A \text{ tower}$$
$$= 0.00421 \text{ for type } B \text{ tower.}$$

Letting

$$k_{10} = 1.151\gamma m_0 g_0 \delta$$

and solving Eq. (587) for E_n in terms of d_s, there results

$$\begin{aligned} E_n &= d_s k_{10}[2 + 57.7\times10^{-5}k_9k_{10} \pm \sqrt{2 + 57.7\times10^{-5}k_9k_{10} - 4.11}] \\ &= d_s U \end{aligned} \tag{588}$$

TABLE 23

$m_0\gamma\delta$	Tower type A		Tower type B	
	U_A	Difference	U_B	Difference
0.35	53,510		51,330	
0.40	62,540	9,030	59 950	8,620
0 45	71,830	9,290	68,790	8,840
0 50	81,320	9,490	77,780	8,990
0 55	91,030	9,710	86,990	9,210
0 60	100,930	9,900	96,340	9,350
0 65	111,070	10,140	105 890	9,550
0 70	121,390	10,320	115,590	9,700
0 75	131,910	10,520	125,460	9,870
0 80	142,620	10,710	135,470	10,010
0 85	153,520	10,900	145,680	10,210
0 90	164,580	11,060	156,020	10,340

where

$$\left.\begin{aligned} U_A &= k_{10}[2 + 3.13 \times 10^{-6}k_{10} + \\ &\quad \sqrt{(2 + 3.13 \times 10^{-6}k_{10})^2 - 4.11}] \text{ for type } A \text{ tower} \\ U_B &= k_{10}[2 + 2.43 \times 10^{-6}k_{10} + \\ &\quad \sqrt{(2 + 2.43 \times 10^{-6}k_{10})^2 - 4.11}] \text{ for type } B \text{ tower} \end{aligned}\right\} \quad (589)$$

Equation (589) furnishes the proportionality factors relating the most economical transmission voltage and the conductor diameter for the types of towers in question. By means of Eq (589), values of U were computed for various assumed values of the product $m_0\gamma\delta$, as listed in Table 23. To find U for any values of m_0, γ and δ, all that is required is to enter the table with the product $m_0\gamma\delta$ and select the corresponding value of U. Values

TABLE 24—ALTITUDE CORRECTION FACTOR

Altitude in feet	δ	Altitude in feet	δ
0	1 00	5,000	0 82
500	0 98	6,000	0 79
1,000	0 96	7,000	0 77
1,500	0 94	8,000	0 74
2,000	0 92	9,000	0 71
2,500	0 91	10,000	0 68
3,000	0 89	12,000	0 63
4,000	0 86	14,000	0 58

of the altitude correction factor δ, as given in the standard Handbook, are found in Table 24.

f. Reactance and Susceptance per Mile of Line Relatively Independent of Conductor Diameter and Line Voltage.—For transmission lines which are so constructed that the separation between conductors is a straight line function of the voltage, as for example in Eq (580), and for which the impressed line voltage is proportional to the conductor diameter, as is assumed in the present discussion (Eq. (588), the inductive reactance and the susceptance per mile of line at a given frequency are both practically independent of the diameter of the conductor used, provided the roughness factor m_0 and the altitude factor δ do not vary, and that the operating voltage is always a constant percentage of the critical disruptive value.

The truth of the above statement is readily verified. Let it be assumed that, for all lines under consideration, the voltage chosen is always 90 per cent of the critical value for the conductor, that the altitude is 1,000 ft., whence $\delta = 0.96$, and that the roughness factor is constant and equal to 0.83. Then

$$\begin{aligned} m_0\gamma\delta &= 0.9 \times 0.96 \times 0.83 \\ &= 0.717. \end{aligned}$$

From Table 23, for this value of $m_0\gamma\delta$, $U_A = 125{,}000$ and $U_B = 119{,}000$ in round numbers.

By Eq. (133) the inductance per mile of one conductor is

$$L = \left(741.13 \log_{10} \frac{D'}{r} + 80.47\right) \times 10^{-6} \text{ henries}$$

and by Eq. (213) the capacitance per mile of one conductor to neutral is

$$C = \frac{0.03883 \times 10^{-6}}{\log_{10} \frac{D'}{r}} \text{ farads.}$$

The corresponding values of inductive reactance and susceptance at 60 cycles are

$$x = 377L$$

and

$$b = 377C.$$

For towers designated as types A and B, respectively, the spacings in inches are given by Eq. (580) as

$$D = 0.00215E_n \quad \text{for type } A$$

and

$$D = 0.00170E_n \quad \text{for type } B$$

whence, for the spacings designated, the equivalent equilateral spacings, by Eq (582) are

$$D' = 1.260 \quad \text{for type } A$$

and

$$D' = 1.238 \quad \text{for type } B$$

or, in terms of the impressed voltage to neutral,

$$\left.\begin{aligned} D' &= 1.260 \times 0.00215E_n \\ &= 0.002709E_n \qquad \text{for type } A \\ \text{and} \qquad & \\ D' &= 1.238 \times 0.00170E_n \\ &= 0.002105E_n \qquad \text{for type } B \end{aligned}\right\} \tag{590}$$

By Eq. (588),

$$E_n = d_sU = 2rU$$

whence

$$\left.\begin{aligned} r_A &= \frac{E_n}{2U_A} \qquad \text{for type } A \\ \text{and} \qquad & \\ r_B &= \frac{E_n}{2U_B} \qquad \text{for type } B \end{aligned}\right\} \tag{591}$$

Substituting the values of U_A and U_B, for the assumed conditions in Eq. (591), and the values of r and D' from Eqs (590 and (591) in the appropriate equations for x and b yields

$$\left.\begin{aligned} x_A &= 377 \times 10^{-6} (741.13 \log_{10} 677.25 + 80.47) \\ &= 0.821 \text{ ohm per mile at 60 cycles for type } A \\ \text{and} \qquad & \\ x_B &= 377 \times 10^{-6} (741.13 \log_{10} 500.99 + 80.47) \\ &= 0.784 \text{ ohm per mile at 60 cycles for type } B \end{aligned}\right\} \tag{592}$$

Similarly,

$$\left.\begin{aligned} b_A &= \frac{377 \times 0.03883 \times 10^{-6}}{\log_{10} 677.25} \\ &= 5.17 \times 10^{-6} \text{ mho per mile at 60 cycles for type } A \\ \text{and} \qquad & \\ b_B &= \frac{377 \times 0.03883 \times 10^{-6}}{\log_{10} 500.99} \\ &= 5.42 \times 10^{-6} \text{ mho per mile at 60 cycles for type } B \end{aligned}\right\} \tag{593}$$

For 25-cycle lines the corresponding values of x and b are

$$\left.\begin{aligned} x_A &= 0.342 \\ x_B &= 0.327 \end{aligned}\right\} \text{ohm per mile at 25 cycles for type } A \qquad (594)$$

and

$$\left.\begin{aligned} b_A &= 2.15 \times 10^{-6} \\ b_B &= 2.26 \times 10^{-6} \end{aligned}\right\} \text{mho per mile at 25 cycles for type } B \qquad (595)$$

While in practice the spacings will not always be those here assumed, yet for high-voltage lines they will usually not deviate greatly therefrom, and hence, for preliminary calculations, the values of x and b here given are quite satisfactory.

j. Derived Line Constants.—For purposes of estimating, and to serve as a check upon calculated values of the derived line constants a_1, a_2, b_1, b_2, c_1 and c_2, tables of these constants have been worked out for 60-cycle lines up to 500 miles for each of two conductor materials, namely, aluminum and copper, and for both types A and B construction. These tables are based upon the values of x and b derived above and upon values of resistance estimated from the conductor diameters and appropriate resistivities. The tables are found in Appendix D. It is of interest to observe that the constants a_1, b_2 and c_2 are roughly proportional to the length of the line and are relatively independent of the kind of conductor material, and almost entirely independent of the conductor area, for the larger sizes of conductors especially.

Tower Cost and Kelvin's Law.—In order to write a complete mathematical expression for the economic relations involved in the generalized statement of Kelvin's law it is necessary to analyze the cost of every item of expense incurred in the construction of the transmission line, and to pick out those which are a function of the conductor area. The annual charge against the latter are then to be expressed as a function of conductor diameter, if possible, and this expression is to be incorporated as a part of the general statement of the law. Since the cost of the transmission-line towers is an important one of these cost items, it will be investigated with this end in view.

From Chap. XII on the Economics of Span Design, the validity of the following statements is apparent.

1. For a given material and a given cable diameter, there is only one tower spacing which yields minimum cost of tower. This has been called the most economical tower spacing. (596)

2. The most economical tower spacing is a function of the cable diameter, as shown by Eq. (566). (597)

3. The most economical tower spacing is different for different conductor materials as shown by the curves, Fig. 73 and Eq. (566). (598)

4. The most economical height h_e of conductor support, is independent of either cable diameter or conductor material for a given ratio $k_3 \div k_2$, as shown on page 231. (599)

The above statements are important considerations in the analysis which follows, and are here set down for future reference.

Items of Tower Cost Analyzed.—The principal factors affecting the cost of towers are:

1. *Conductor Tension, and Wind and Snow Loads.*—The tower must be designed to withstand the maximum conductor tension plus wind loads on its projected area acting horizontally, together with its own dead weight and assumed snow loads acting vertically. Of these loads, the tension is the principal one, and it is proportional to the conductor area, that is, to d_s^2.

2. *Line Voltage.*—As already pointed out, if a given factor of safety is used, the spacing of transmission line conductors is proportional to the line voltage The effect of spacing upon cost is not a linear relation, however. Just as tower cost is proportional to the square of its height, so is a portion of its cost also proportional to the square of its spread. Hence, two towers, designed for the same tensions and the same heights, will yet differ in costs if designed for two different voltages, the higher voltage requiring the greater cost. Considering the insulators as a part of the line supports, this item too is a function of the line voltage. Since, however, for most economical design the voltage is a function of d_s, that part of the tower cost, including insulators and their hardware, which varies with the line voltage must be a function of d_s. For the reasons given above, this cost varies in proportion to d_s^n, where n is some number between 1 and 2.

3. *Conductor Height.*—For a given voltage and a given cable tension, the turning moment, about the base of the tower, is proportional to the tower height. Hence, the area of the individual member is proportional to the height, and the weight of the tower as a whole is proportional to the square of its height.

The items into which the total cost of towers was subdivided, it will be recalled, are

1. Cost of tower at place of erection.
2. Cost of erection of tower.
3. Cost of tower site.
4. Cost of foundation installed.
5. Cost of location and inspection of support.
6. Cost of insulators at location of tower.
7. Cost of placing insulators and cable.

These itemized costs may now be used in evaluating the influence of conductor diameter upon the cost of towers The variation in cost due to conductor tension and line voltage will first be considered.

Influence of Voltage on Cost as a Function of d_s.—Since neither voltage nor tension greatly affects either items 3, 5 or 7, and since they are small and relatively unimportant in any case, only items 1, 2, 4 and 6 are here considered.

In order to evaluate the sum of these cost items as a function of the line voltage, the following procedure is employed:

Obtain from the manufacturer bids on both anchor and suspension towers of the types under consideration, as for example the single-circuit towers of type *A* and the double-circuit of type *B*. Bids should be requested for each type of tower designed for each of three different voltages as, for example, 100, 150 and 200 kv., all towers to be of the *same height*, say 45 ft., and all to withstand the *same tension* per cable, say 6,000 lb. In addition to the above, two more sets of quotations should be obtained on similar towers of the same height (45 ft.), both designed for one of the above voltages, say of 150 kv., but built to withstand different tensions, say of 3,000 lb. and 9,000 lb. These data will serve as the basis for finding the desired relation of conductor diameter to voltage and tension.

The conductor spacings, for towers of type *A* and for the above voltages, may be read directly from Fig. 76. The same figure also specifies conductor spacings and crossarm dimensions for towers of type *B* if general proportions as in Fig. 75 are used.

The expression "height of tower," as here used, means the elevation above ground level of the point of support of the lowest conductors or it is the minimum allowable clearance to ground plus the maximum deflection of the critical catenary. For an anchor tower, it is the elevation above ground of the

lowest crossarm, and, for a suspension tower, it is the elevation above ground of the lowest crossarm minus the length of the insulator string.

The number of insulators per string may be found by dividing the line voltage by the allowable volts per disc; or, approximately,

$$\text{Number of standard suspension discs} = \frac{\text{Line voltage}}{17,000} \tag{600}$$

$$\left.\begin{array}{l} \text{Number of discs for 100 kv.} = 6 \\ \text{Number of discs for 150 kv.} = 9 \\ \text{Number of discs for 200 kv.} = 12 \end{array}\right\} \text{For length of insulator string, see Fig 75.}$$

Thus the manufacturer can be given full specifications for overall tower dimensions for both anchor and suspension towers of both types A and B for an assumed height of 45 ft. and a tension of 6,000 lb.

Assuming that the most economical voltage will be used, it is expressable as a function of d_s. From Table 23, for an elevation of 1,000 ft. above sea level, for example, if the voltage is approximately 90 per cent of the corona forming value for the conductor used, the potential difference between conductors is

$$E = 215,000d_s \text{ for type } A \text{ tower}$$
$$E = 205,000d_s \text{ for type } B \text{ tower.}$$

$$\left.\begin{array}{l} \text{From this relation, for type } A \text{ towers,} \\ \quad 100,000 \text{ volts corresponds to } d_s = 0.465 \text{ in.} \\ \quad 150,000 \text{ volts corresponds to } d_s = 0.698 \text{ in.} \\ \quad 200,000 \text{ volts corresponds to } d_s = 0.930 \text{ in.} \\ \text{and, for type } B \text{ tower,} \\ \quad 100,000 \text{ volts corresponds to } d_s = 0.487 \text{ in.} \\ \quad 150,000 \text{ volts corresponds to } d_s = 0.731 \text{ in.} \\ \quad 200,000 \text{ volts corresponds to } d_s = 0.975 \text{ in.} \end{array}\right\} \tag{601}$$

It should be noted here that the number of insulators for an anchor tower is at least twice the number used on a suspension tower. It is more than twice that number if high-strength conductors are used in such a way that the mechanical strength of one standard string is not sufficient to carry the maximum conductor tension. In that case, two insulator strings are placed in

parallel. Due to the decreased dielectric strength of a parallel group of insulators below that of a single string, additional units must be added in series, such that a string of 12 insulators is increased to 14 when connected in parallel with another string. For preliminary calculations the assumption that the anchor towers each require twice the number of insulators required by each suspension tower, may be used. After the conductor size is finally found, more refined calculations should be made for finding the insulator requirements.

After the cost data are obtained from the manufacturer and before final tabulation of the separate items is made as in Table 32 page 343, all items of cost except 3, 5 and 7 are recalculated for the *average* support *per line.* This is done by using the relation

Cost item for average support

$$= \frac{\text{Cost item for anchor tower} + \frac{T_s}{T_a}\text{ cost item for suspension tower}}{1 + \frac{T_s}{T_a}} \tag{602}$$

where T_s = total number of suspension towers per line.
T_a = total number of anchor towers per line.

Item 6 should be taken to include the insulator hardware, as well as insulators.

The cost of the *average line support* for each type of tower investigated is thus determined, (*a*) For each of three different voltages and a fixed tension; (*b*) for each of three different tensions and a fixed voltage.

For the two circuit towers of type *B*, the total cost is divided by two to get the cost per average support *per line.* From the relations given in Eq. (601), the conductor diameter d_s, corresponding to each price quoted, is available. The values d_s^2 obtained from Eq. (601) are plotted as abscissas against the sum of items 1, 2, 4 and 6 for the *average tower per line* as ordinates, and a straight line is drawn through the three points thus located. A straight line may not touch all three points, but since it is desired to express the cost as a function of d_s^2, and it is known that the cost is proportional to d_s^n, where n does not differ greatly from 2, it is apparent that the quadratic law can be made to fit the actual curve fairly well over a limited range at least. Hence,

the straight line is drawn to coincide with a smooth curve through the three points *in the region of the values of d_s, within which the value of d_s for the project in question is most likely to fall.* In this way the quadratic law will fairly well fit the curve over the desired range, and the degree of accuracy over the remainder of the curve is immaterial.

The straight line thus drawn is equivalent to the parabola that would result were the cost plotted against the first power of d_s. The equation of the straight line is

$$\text{Cost items } 1 + 2 + 4 + 6 \text{ of average support} = k_v + md_s^2 \quad (603)$$

where k_v = that part of items 1 + 2 + 4 + 6 which is independent of the line voltage, or conductor diameter, and is the intercept of the straight line on the cost axis. The constant m is the slope of the straight line, in terms of cost and d_s^2.

The above analysis shows how the influence of transmission voltage upon the cost of towers may be found. The constant m, it should be noted, is independent of the tension and of the height of tower, since the tension T_1 and the height h_e are constant factors in the three costs used to locate the straight line. Thus m has the same value whether the tension is 5,000 lb. or 8,000 lb. per conductor; it depends only upon the line voltage. If quotations were obtained for the same three voltages but for each of two other constant tensions T_2 and T_3, the curves of cost *vs.* d_s^2 for the latter would be straight lines parallel to the curve for T_1. Thus, in Fig. 77, if T_1 is the cost curve for tension T_1 = 6,000 lb., T_2 and T_3 would represent the corresponding curves for T_2 = 3,000 lb. and T_3 = 9,000 lb. Since it is known that these lines are parallel it is unnecessary to have more than one point on each, once the slope is known. This point is obtained for each line from the quotations for towers built for the two additional tensions but for the same voltage as one of those specified for the data of curve T_1 as already explained.

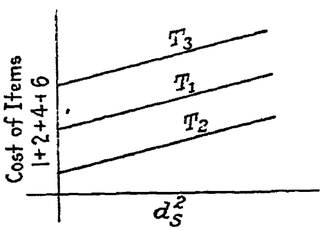

Fig. 77.—Tower costs as functions of conductor tensions.

Influence of Tension on Cost as a Function of d_s.—The influence of tension on cost may now readily be found from the data already assembled, as soon as the relation between conductor area and tension is known for each of the conductor materials to be investigated.

The elastic limit, assuming stranded conductors, for aluminum and copper may be assumed in accordance with the data given in Chap. II as

$$\left.\begin{array}{lll} \text{Aluminum} & T_e = 14{,}000 \text{ lb. per sq. inch} \\ \text{Copper} & T_e = 28{,}000 \text{ lb. per sq. inch} \\ \text{Steel} & T_e = 54{,}000 \text{ lb. per sq. inch} \end{array}\right\} \qquad (604)$$

If it be assumed that the conductor may not be safely stressed beyond 75 per cent of its elastic limit, and that, for the sizes of cable considered, the ratio of $d_s \div d$ may be taken as 1 151, then the relation between the maximum allowable tension T_m and outside cable diameter d_s is

$$\left.\begin{aligned} d_s^2 &= \frac{4 \times (1.151)^2}{\pi} \times \frac{T_m}{0.75 \times 14{,}000} \\ &= 16.1 \times 10^{-5} T_m \text{ for aluminum} \\ d_s^2 &= \frac{4 \times (1\ 151)^2}{\pi} \times \frac{T_m}{0.75 \times 28{,}000} \\ &= 8.04 \times 10^{-5} T_m \text{ for copper} \\ d_s^2 &= \frac{4 \times (1.151)^2}{\pi} \times \frac{T_m}{0.75 \times 54{,}000} \\ &= 4.17 \times 10^{-5} T_m \text{ for steel} \end{aligned}\right\} \qquad (605)$$

If, as is sometimes done, it is assumed that the conductor may be safely stressed up to its elastic limit, on the theory that, if so stressed, the conductor will simply stretch slightly to relieve the tension without injuring it, then the above figures may be increased by 33 per cent. The exact values to be used will vary somewhat with the judgment of the designing engineer who is responsible for the work. We are here concerned mainly with the principle involved rather than with the precise values to be used.

In Eq. (605), for T_m may be substituted in succession the three values of tension for which bids on towers have already been obtained. These substitutions will yield the three corresponding values of conductor diameters for which these tensions are permissible, or the desired values of d_s^2.

Referring now to Eq. (603) and Fig. 77, it is apparent that each tension curve has a separate value of the intercept k_v, while the slope is the same for all curves. Thus, k_v varies with the tension but not with the voltage. From the three curves of Fig. 77, the law of variation of k_v, with tension or with d_s^2 may be

found by evaluating k_v for each of the three curves and replotting these values, again using k_v (costs) as ordinates and d_s^2 as abscissas From the argument already presented, it is apparent that these values of k_v are approximately proportional to d_s^2. Hence, the new plot will yield a straight line as before, whose equation is

$$k_v = k_T + nd_s^2 \tag{606}$$

where k_T is the intercept on the k_v (cost) axis and n is the slope of the new curve. By substituting the value of k_v from Eq. (606) in Eq (603), the cost of items 1 + 2 + 4 + 6, for the average support and as a function of conductor area, is found to be

$$\text{Cost of items } 1 + 2 + 4 + 6 = k_T + (m + n)d_s^2. \tag{607}$$

Evidently, k_T is the same for all conductor materials, since it is independent of T_m, whereas the slope n, which is a function of the tension, has a different value for each material. Thus Eq (607) takes account of the variation in tower cost which results from variations in voltage and tension. It must be remembered, however, that it was derived on the assumption of a *fixed height* of tower. The influence, of tower height on the cost of tower, will now be considered to determine how, if at all, it should enter into consideration from the standpoint of economics

Influence of Tower Height on Cost as a Function of d_s^2.—According to item 4 Eq. (599), the height of tower is fixed for a given ratio of $k_3 \div k_2$, no matter what the size of d_s or what the material of the conductor. By definition (Eq (557)), for type A tower,

$$\frac{k_3}{k_2} = \frac{(\text{Items } 4 + 5 + 6 + 7)h^2}{(\text{Items } 1 + 2 + 3)}. \tag{608}$$

An examination of these items and the way in which they are influenced by conductor diameter shows that the effect of cable diameter on the ratio $k_3 \div k_2$ is very small since the items in the numerator and those in the denominator of Eq. (608), with the exceptions of the small items 5 and 7, are similarly affected. Since 4 + 6 is not greatly different in size from 4 + 5 + 6 + 7, the effect of the two small items 5 and 7 becomes quite negligible. Hence, considering the ratio practically independent of conductor diameter, and since the height of tower is constant for a given ratio, regardless of conductor size or material, the *height of tower does not enter as a factor to be considered in the economics of transmission-line design.*

For the double-circuit towers of Type B, Eq. (608) needs slight modification. This tower carries its circuits in a vertical plane, and, from the standpoint of turning moments about the foundation, the pull of the line conductors may be thought of as being concentrated in a nearly horizontal plane passing through the point of conductor attachment on the middle crossarm. Therefore, the cost of the tower structure is not proportional to the square of its height measured to the lower crossarm, but rather to the square of its height measured to the maximum point on the middle conductor catenary. Hence, when using the general proportions of tower and the spacings, as shown in Fig. 75, the correct equations corresponding to Eq. (608) for the two types of towers are

$$\left.\begin{aligned} \frac{k_3}{k_2} &= \frac{h^2(\text{Items } 4 + 5 + 6 + 7)}{(\text{Items } 1 + 2 + 3)} \text{ for type } A \text{ tower} \\ \frac{k_3}{k_2} &= \frac{\left(h + \frac{D_v}{12}\right)^2(\text{Items } 4 + 5 + 6 + 7)}{(\text{Items } 1 + 2 + 3)} \text{ for type } B \text{ tower} \end{aligned}\right\} \quad (609)$$

(For D_v see Fig. 75 and page 255.)

For type A tower the value of k_1, the minimum clearance to ground, is assumed or specified by law. For the double-circuit tower, since the height is measured to the point of attachment of the middle conductor, a value k_1' is used in place of k_1 where

$$k_1' = k_1 + D_v. \quad (610)$$

For example, for a 150 kv. line whose minimum clearance to ground is to be 28 ft., by Fig. 75 and page 255,

$$D_v = \frac{3D}{\sqrt{10}} = 0.947D$$

and by Eq (578),

$$D = 0.0017E_n$$

whence

$$\begin{aligned} D_v &= 0.947 \times 0.0017 \times 150{,}000 \div \sqrt{3} \\ &= 138 \text{ inches} \\ &= 11.5 \text{ feet} \end{aligned}$$

and

$$k' = 28 + 11.5 = 39.5 \text{ ft.}$$

Equation for Total Cost of Towers.—The total cost of the average line support may now be evaluated. The cost, as

influenced by tension and voltage, has already been expressed as a function of d_s^2 by Eq. (607). The height of tower, as shown above, is not a function of d_s^2. The height of tower does affect the cost of certain items in proportion to its square, however The items thus affected are 1, 2 and 3. The cost of the average support should therefore be corrected by an amount equal to the difference between the assumed value of items 1 + 2 + 3 and the value found from the equation

$$\text{Corrected cost items } 1 + 2 + 3 = \frac{h_e^2}{h^2} \times (\text{Items } 1 + 2 + 3) \quad (611)$$

where

h_e = the most economical tower height

and

h = actual tower height for which costs were obtained.

The difference between cost items 1 + 2 + 3, as per Eq (611), and the assumed cost of these items is the correction factor

$$k_4 = (\text{Items } 1 + 2 + 3)\left(\frac{h_e^2}{h^2} - 1\right) \quad (612)$$

Items 5 and 7 are independent of voltage, tension and height, and may be represented by the constant k_5; that is,

$$k_5 = \text{Items } 5 + 7 \quad (613)$$

Combining Eqs (607), (612) and (613), the total cost of the average line support may be written

$$\text{Total cost of average line support} = (m + n)d_s^2 + k \quad (614)$$

where

$$k = k_T + k_4 + k_5. \quad (615)$$

Total Cost of Line Supports.—From Eq (615), the total cost of line supports may be written, for

L = total length of line in feet

and, since S = length of the most economical span in feet

$$\text{Total cost of line supports} = \frac{L}{S}[(m + n)d_s^2 + k]. \quad (616)$$

By Eq. (566), however,

$$S = \frac{k_7(d_s + k_6) - k_8}{d_s + k_6}.$$

Substituting this value of S in Eq. (616) yields the equation of tower cost as a function of conductor diameter. It is

$$\text{Total cost of line supports} = \frac{L[(m + n)d^2 + k](d_s + k_6)}{k_7(d_s + k_6) - k_8}. \tag{617}$$

The meaning of the constants in Eq. (617) are summarized below for convenience. They are

$$\left.\begin{array}{rl} m = & \text{slope of cost } vs.\ d_s^2\text{, from curves of Eq. (603).} \\ n = & \text{slope of cost } vs.\ d_s^2\text{, from curves of Eq. (606).} \\ k_7 = & \text{constant of Eq. (565), found from Plates II and IV,} \\ & \text{for aluminum and copper respectively.} \\ k_8 = & \text{constant of Eq. (565), found from Plates III and V,} \\ & \text{for aluminum and copper, respectively.} \\ k_6 = & \text{constant of Eq. (565), having the following} \\ & \text{values:} \end{array}\right\} \tag{618}$$

$$k_6 = +0.20 \text{ for aluminum}$$
$$k_6 = -0.15 \text{ for copper}$$
$$k_6 = -0\ 30 \text{ for steel.}$$

Equation (617) is too complex in form to be of much use as an item to be incorporated in a general statement of Kelvin's law. It was found that, for ranges of d_s between 0.6 and 2 in. and for values of k_1 and $k_3 \div k_2$ lying within the limits of Plates II to V, inclusive, the quantity

$$\frac{[(m + n)d_s^2 + k](d_s + k_6)}{k_7(d_s + k_6) - k_8}, \tag{619}$$

which is the cost of supports per foot of line, is very closely approximated over a considerable range by the simpler equation of the parabola

$$Md_s^2 + N \tag{620}$$

where M and N are constants.

The constants are found by computing the value of Eq. (619) for a number of values of d_s, and the required values of $k_3 \div k_2$ and k_1, and plotting the results on coordinate paper, using values from Eq. (619) as ordinates against d_s^2 as abscissas. The resulting graph is approximately a straight line. Again, the line should be drawn so as to fit the curve best over the range of values within which the looked-for value of d_s lies. The slope of this

straight line is M, and its intercept on the cost axis is N. The total cost of line supports then finally becomes

$$\text{Cost of line supports} = L(Md_s^2 + N). \tag{621}$$

Tower Cost as a Factor in Kelvin's Law.—Kelvin's law embraces all factors of transmission-line cost which vary with the conductor area. The purpose of the above analysis of tower costs is to derive an expression in which tower costs are given as a function of conductor area, or d_s^2, in order that that portion of such cost, which varies with the conductor area, may be taken into proper account in a mathematical statement of the law of economy. It is to be remembered also that three items or groups of items constitute the cost factors involved, namely, (*a*) the line conductors and ground cables; (*b*) line supports or towers; (*c*) high-tension transformers and other terminal apparatus.

Of these three items, the costs of (*a*) and (*c*) can be very accurately expressed as a function of the conductor area. The evaluation of (*b*) is far more difficult and the results may be somewhat less satisfactory. Since it is only one of three items, however, and by no means the most important one, unavoidable errors or uncertainties that may be involved will not seriously affect the result. As a matter of fact, a good first approximation of conductor diameter is obtained by a solution of Eq (635) neglecting the tower factor entirely. The value of d_s thus obtained will of course be larger than the correct value

In Eq (621) LMd_s^2 is that portion of the total cost of towers which varies with the conductor area, while the part LN is independent of conductor area. Thus,

$$LMd_s^2 \tag{622}$$

is that part of the cost of line supports which must be embraced by Kelvin's law, and the yearly charge against this item, in interest and depreciation, must be added to the fixed charges against the line conductors as given by Eq. (524). The annual fixed charges against the line supports then is

$$\text{Tower fixed charges } \frac{p_2 LMd_s^2}{100} \tag{623}$$

where p_2 is the per cent interest and depreciation chargeable to towers, expressed as a whole number.

Cost of Ground Cables.—The function of the ground cables suspended from tower to tower above long transmission lines is to

provide protection against lightning surges and to transfer stresses to adjacent towers in case of the mechanical failure of a line conductor. These cables are generally made of steel and are clamped to the tower structure at its highest point so that this grounded steel catenary is situated well above the transmission conductors. For type *A* tower, if one ground wire is used, its location should be midway between any pair of conductors, so that the ground cable catenary may have the maximum clearance to the conductors. The best construction, however, from a mechanical standpoint, is the use of two ground wires as shown in Fig. 75. This minimizes eccentric stresses upon the tower structure. For type *B* tower only one ground cable is used, which is clamped to the apex of the tower top, as shown in Fig 75. Thus, one ground wire is sufficient for two transmission lines of the type *B* construction, giving adequate protection electrically, whereas two ground wires should be used for each transmission line of type *A* construction.

From the standpoint of electrical protection, a very small ground wire would be sufficient and its size would largely be determined by analysis of its mechanical loading at minimum temperature and at maximum sleet and wind load for the span in question; but if its duty is also to increase the rigidity of the suspension towers by transferring stresses to the anchor towers, in case of conductor failures, its size should be ample to not only support its own weight and normal loadings, but to also transmit the stresses upon the tower caused by the failure of at least one line conductor.

Accordingly, since the cost of the ground cable is a function of the size of conductor, this cost should be estimated in terms of conductor area and should be included as an item in the mathematical statement of a generalized Kelvin's law. By inspection of Eq. (605) the reader will notice that the maximum allowable stresses for aluminum, copper and steel are in the proportion 1, 2 4, respectively. This simple relationship is responsible for the following assumptions, which, although rather arbitrary, may form a convenient basis for the determination of the proper ground cable sizes to be used for long transmission lines, and an analysis of their cost as a function of conductor area.

Assumption 1.—Steel cables when used as conductors in transmission circuits will be of such size that a mechanical failure of the line may be considered beyond the range of possibility.

Hence, the ground cable may be made of the same size as the conductors proper, and it may be strung with the same maximum catenary deflection at the center of the span as the conductors

Assumption 2.—Copper cables when used as conductors are to be considered within the range of possible mechanical failure. Hence, if the ground wire be made of *the same* metallic cross-sectional area as that of the copper conductor, and if the ground cable be strung with a maximum catenary deflection equal to that of the conductors, the ground cable will be of sufficient strength to transmit to the anchor towers the stresses due to the failure of one conductor. This follows by reason of the fact that the steel ground cable is twice as strong as the copper and will therefore be able to carry its own dead weight and loading plus that of a copper cable of equal area.

Assumption 3.—Aluminum cables when used as conductors, are to be considered within the range of possible mechanical failure. Hence, if the ground wire be made of *half* the metallic cross-sectional area as compared to the aluminum conductor, and if the ground cable be strung with a maximum catenary deflection equal to that of the conductors, it will be of sufficient strength to transmit to the anchor towers the stresses due to the failure of one conductor.

Since the specific gravity of aluminum is less than half that of steel, the weight per unit length of catenary under extreme sleet and wind conditions at minimum temperature is approximately the same as that of a steel cable of half the cross-sectional metallic area. This fact lends additional support to assumption 3. The specific gravities of copper and steel are near enough alike to also strengthen assumption 2. The catenaries formed by the conductors and ground cables, if the sizes specified by the above assumptions are used, will, therefore, perform very nearly alike over the range of weather conditions and loadings to which the transmission line is exposed in a given locality. This will always insure proper clearances of conductors to ground, and the factors of mechanical safety will be practically alike for aluminum, copper and steel of both types *A* and *B* tower construction, thus permitting a fair cost comparison between these materials and tower types to be made.

If two ground cables are used for type *A* tower construction, the combined metallic cross-sectional area of the two cables should be equal to that of the single ground cable for type *A* or *B* towers.

According to the foregoing assumptions the fixed charge against the ground wire, per line (Eqs. (574), (575), and (576)) is

Ground cable fixed charge

$$\left.\begin{aligned}&= 2.05 \times 10^{-5} d_s^2 L p_{Fe} B_{Fe} W_{Fe} && \text{for aluminum}\\ &= 4.10 \times 10^{-5} d_s^2 L p_{Fe} B_{Fe} W_{Fe} && \text{for copper and steel}\\ & && \text{for type } A \text{ towers}\\ &= 1.03 \times 10^{-5} d_s^2 L p_{Fe} B_{Fe} W_{Fe} && \text{for aluminum}\\ &= 2.05 \times 10^{-5} d_s^2 L p_{Fe} B_{Fe} W_{Fe} && \text{for copper and steel}\\ & && \text{for type } B \text{ towers}\end{aligned}\right\} \qquad (624)$$

Including in the conductor fixed charge that of the ground cable, Eq. (576) becomes

Conductor fixed charge

$$= 12\,31 \times 10^{-5}\, d_s^2 L(p_1 BW + gP_{Fe} B_{Fe} W_{Fe}) \qquad (625)$$

where

p_{Fe} = per cent interest and depreciation chargeable against the ground cable.

B_{Fe} = cost of steel cable in dollars per pound.

W_{Fe} = weight of steel in pounds per cubic foot

$$\left.\begin{aligned}g &= 0.167 \text{ for aluminum conductors}\\ &= 0.333 \text{ for copper and steel conductors}\end{aligned}\right\} \text{type } A. \text{ tower.}$$

$$\left.\begin{aligned}&= 0.083 \text{ for aluminum conductors}\\ &= 0.167 \text{ for copper and steel conductors}\end{aligned}\right\} \text{type } B \text{ tower.}$$

High-tension Apparatus and Housing.—Owing to the greater amount of insulation required, the greater clearances necessary and the increased difficulty of manufacture, etc., the cost of high-tension apparatus increases rapidly with the operating voltage This fact has an important bearing upon the choice of the most economical transmission voltage, and results, in the selection of a lower voltage than would be used, were these costs independent of the voltage. It is important also to observe that *the total cost of the high-tension apparatus, required for a given project, depends only upon the kilovolt-amperes developed, and the voltage used, and is independent of the distance over which the power is to be transmitted.* Accordingly, for a very short line, this item will have a very important influence upon the choice of the conductor diameter and transmission voltage, since it represents a relatively large percentage of all cost items which influence their choice, while in a very long line the cost of terminal apparatus will be a relatively

smaller part of these costs and will accordingly influence the choice of conductor much less. Expressed in another way, it may be stated that the extent to which these cost items bear upon the selection of the most economical voltage and conductor diameter, is measured by the cost of high-tension equipment per mile of line, or is in inverse proportion to the length of line.

In order to give proper weight to the cost of terminal apparatus, as influencing the choice of conductor diameter and line voltage, the variation of costs of the required equipment is studied for a suitable range of line voltages within which the voltage of the project in question is known to lie. That is, the combined cost of all items of such apparatus is obtained from the manufacturer for each, of say three voltages, covering the desired range and for the type of station layout to be used. Kelvin's law, it will be recalled, is concerned only with such cost items as are a function of the conductor area, *i.e.*, a function of the diameter squared. Furthermore, it has already been shown that the relation which should hold between the conductor diameter and line voltage is given by the constant U, of Table 23. We therefore have a means of expressing the cost of terminal apparatus as a function of d_s, and may introduce the variable part of this cost into a mathematical statement of Kelvin's law.

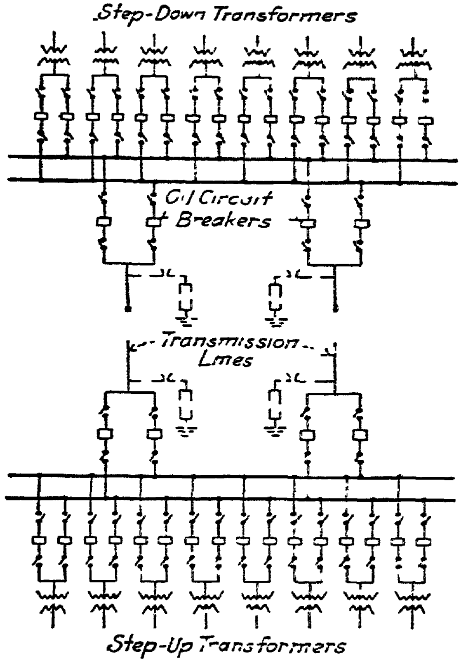

FIG. 78.—Transmission line and station wiring diagram.

Method.—The procedure, suggested for formulating the law of variation of terminal apparatus and housing, as a function of the line voltage, is as follows: Make up a circuit diagram of the high-tension apparatus required for the project, as illustrated in Fig. 78. Most station layouts will be simpler and less costly than the one represented. Whatever type of layout is proposed

should form the basis of this investigation, however Other types of layouts are shown in Figs. 80 to 83, inclusive. Since the size of the development is known to begin with, the number and rating of each unit is known, or may be decided upon, irrespective of the number of transmission lines, with the exception of the line circuit breakers, disconnecting switches and lightning arresters, if any. The ratings and number of line switches and the number of lightning arresters will obviously depend upon the number of transmission circuits. For reasons which will be developed more fully later, it is apparent that in the ultimate development of any project which is economically designed, the separate transmission circuits, between a given pair of generating and receiving stations, will not exceed two in number unless it should happen that the maximum power limit of the two lines should be exceeded by the load to be transmitted. In such a case, more than two lines would be required. Two transmission circuits may therefore be assumed as the number required, unless conditions are such that a single circuit will answer That is, if during times when the line is out of service, either due to failure or for the making of necessary repairs, the load can be economically supplied from other sources until service is re-established, a single line will prove to be more economical than two lines. In such cases a single line should be used.

After having prepared the wiring diagram for the power station and lines, indicating the probable number of lines, a print of this diagram together with other necessary data as to the number of units of each kind, their kilovolt-ampere, rating, etc , is submitted to the manufacturer, with a request for quotations on the required apparatus for each of several standard transmission voltages, covering the range within which the project will fall. These quoted prices are then tabulated and totaled for each of the voltages chosen.

Cost of Wiring, Housing, Etc.—Estimates should next be made of the cost of structures required for the proper housing, supporting and connecting the substation equipment. Under this heading are included the costs of wiring, foundations, supports for buses and other wiring, housing, if any, etc. As with the apparatus itself, so here, for any given voltage these costs will vary considerably with the degree of complexity of the substation layout, and, to some extent, with the total kilovolt-amperes of station capacity.

For a given station layout, however, the total costs of the above items are a function of the line voltage used. This is apparent from the fact that the amount of area, required to properly arrange and support the equipment, is proportional to the square of the line voltage, since, for a given factor of safety, the clearances necessary between transformers, buses, switches, etc., increase with the line voltage in two dimensions. The cost of the structures will therefore vary with E^2.

Total Cost of Substation Equipment, Including Wiring and Housing.—After the estimates of the above items of cost have been made for the chosen type of layout and for each of the assumed voltages under consideration, the totals obtained should be added to the totals for the apparatus itself. Assuming that this has been done, we now have a separate total cost for each of a number of voltages. The next step is to plot these totals as ordinates against the squares of their corresponding voltages. Investigation will show that, through the points thus located, a curve approximating a straight line can be drawn, that is, the total cost varies approximately as the square of the line voltage over a considerable range of voltages (Fig. 79). Accordingly, a parabola of the form

$$C = k'_{11}E^2 + k_{12} \tag{626}$$

may always be found which will closely fit the curve of cost *vs* voltage over the desired range of voltage, where

C = total cost of terminal equipment, wiring, housing, etc.
k_{12} = the intercept of the cost curve on the cost axis, representing that part of the cost which is independent of line voltage.
k'_{11} = that part of the cost which varies with the line voltage and which must be considered in the application of the modified Kelvin's law.
E = the line voltage in volts.

Such a curve of cost *vs.* E^2, for a 50,000-kva. plant of the type shown in the diagram of Fig 78, is the curve in Fig. 79. The costs for this curve were estimated for 88, 110, 132, 154 and 220 kv. Inspection of this figure shows that while the straight line *a* fits the curve fairly well over the entire range of from 88 to 220 kv., yet, if one knew to begin with that the project considered would probably require a voltage of less that 154 kv., it

would be better to use the line b, while, for voltages above 132 kv., the line c is more nearly correct. That is, the straight line should be chosen to fit the particular part of the curve which embraces the voltage of the project in question.

It may be argued that the total cost of terminal apparatus, housing, wiring, etc., will not be correctly represented, over a wide range of voltages, by a parabola of the form given in Eq. (626); and that, therefore, the results obtained by using the variable part ($k'_{11}E^2$) as a factor in Kelvin's law, will be in error.

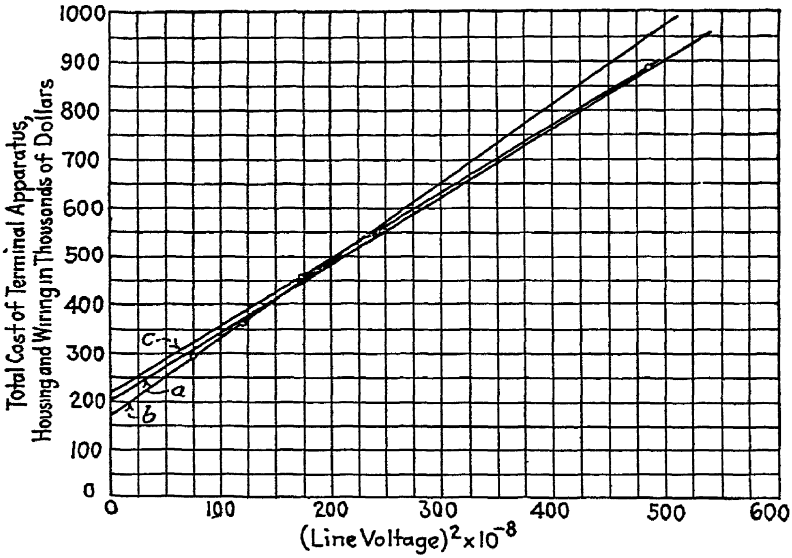

FIG. 79.—Cost estimate for 50,000 kva substation.

It is true that, were the exponent 1.6 used instead of 2 in Eq. (626), the equation would usually fit the curve better over its entire range. It is not necessary for the purpose in hand, however, that the equation should fit the curve throughout its entire length. Close agreement, over a range of sufficient length to include the voltages of the given project, is sufficient. The far greater simplicity, which results, together with the fact that accuracy in making the estimate is in no way sacrificed if one always selects the most suitable portion of the curve, is at once the reason and the justification for selecting 2 instead of 1.6 as the exponent of E in Eq. (626).

Empirical Equation of Cost. Constants k'_{11} and k_{12}.—Equation (626) is the general form of the empirical equation of cost, the

constants of which have to be found. These are to be obtained from the straight line relations of Fig. 79. To the scale of the curve, k_{12} is the amount of the cost intercept while k'_{11} is the slope of the line. Thus, for the entire range of voltages from 88 to 220 kv., using the line a, the constants are:

$$\left.\begin{aligned} k'_{11} &= 14\ 3 \\ k_{12} &= 195{,}000 \end{aligned}\right\} \text{88 to 220 kv.}$$

Fig 80

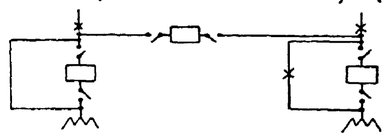

Fig 81.

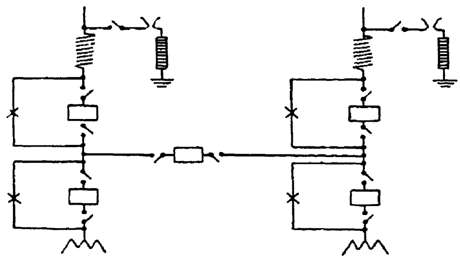

Fig 82

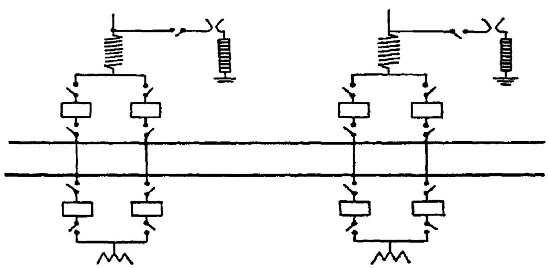

Fig. 83

Figs. 80, 81, 82 and 83.—Substation wiring diagrams of various degrees of complexity

Similarly, for the lines b and c:

$$\left.\begin{aligned} k'_{11} &= 16.2 \\ k_{12} &= 170{,}000 \end{aligned}\right\} \text{88 to 187 kv.}$$

$$\left.\begin{aligned} k'_{11} &= 13\ 6 \\ k_{12} &= 215{,}000 \end{aligned}\right\} \text{132 to 220 kv.}$$

TABLE 25

Type of substation circuit	50,000-kva plant						100,000-kva plant					
	88 to 220 kv.		88 to 187 kv		132 to 220 kv		88 to 220 kv		88 to 187 kv		132 to 220 kv	
	$10^6 k'_{11}$	$10^{-3} k_{12}$	$10^6 k'_{11}$	$10^{-3} k_{12}$	$10^6 k'_{11}$	$10^{-3} k_{12}$	$10^6 k'_{11}$	$10^{-3} k_{12}$	$10^6 k'_{11}$	$10^{-3} k_{12}$	$10^6 k'_{11}$	$10^{-3} k_{12}$
Fig 80	5 9	140	7 1	120	5 5	153	6 0	180	7 7	150	5 4	200
Fig 81	7 5	155	9 2	125	7 1	165	8 5	175	9 9	150	7 8	195
Fig 82	11 1	170	12 3	155	10 3	195	12 1	195	13 2	180	11 1	220
Fig 83	14 3	195	16 2	170	13 6	215	15 3	215	17 2	190	14 6	240

As already pointed out, the constants k'_{11} and k_{12} vary with the degree of complexity of the layout and with the kilovolt-amperes rating of the project. In order to make available approximate values of these constants for purposes of estimating, these constants were calculated from estimated total costs of substation equipment housing, etc., for circuits of four different types, as shown in Figs 80, 81, 82 and 83, and for total capacities of 50,000 and 100,000 kva. The constants for each type and rating are computed for each of three ranges of voltage, as illustrated in connection with Fig. 79 These constants are listed in Table 25.

Cost of Terminal Apparatus and Housing as a Function of Conductor Diameter.—Before the cost of terminal equipment and housing can be introduced as a factor in a mathematical statement of Kelvin's law of economy, the cost must be expressed as a function of conductor diameter. This is readily accomplished, for, by Eq. (588), the voltage to neutral is

$$E_n = d_s U$$

whence

$$\begin{aligned} E^2 &= 3E_n^2 \\ &= 3d_s^2 U^2. \end{aligned}$$

Substituting this value of E^2 in Eq. (626) yields the desired equation of cost in terms of conductor diameter. It is

$$\text{Cost of terminal apparatus and housing} = 3k'_{11}U^2d_s^2 + k_{12}$$
$$= k_{11}d_s^2 + k_{12} \qquad (627)$$

where

$$k_{11} = 3k'_{11}U^2 \qquad (628)$$

The Annual Charge.—The only part of Eq. (627), which influences the choice of conductor diameter and voltage, is the first term of the right-hand member. This item of cost is

$$C' = k_{11}d_s^2 \qquad (629)$$

and the annual charge against this item is

$$\text{Annual charge against terminal apparatus and housing}$$
$$= 0.01p_3k_{11}d_s^2 \qquad (630)$$

where

p_3 = weighted average percentage (expressed as a whole number) applicable to the total cost of terminal apparatus and housing, to take care of the annual interest and depreciation on this item.

Equation (630) may now be incorporated in a mathematical statement of the generalized law of economy.

The final equation, which includes all of the factors to be considered, and which balances the selling value of the annually wasted energy in the line against the annual fixed charges on all items whose costs vary with the conductor diameter or voltage, may now be set down.

Kelvin's Law Involving All Cost Factors.—From Eq. (625), the annual fixed charge against the conductor is

$$12.31 \times 10^{-5}d_s^2L(p_1BW + gp_{Fe}B_{Fe}W_{Fe})$$

and from Eq (623), the annual fixed charge against the line supports is

$$0.01p_2LMd_s^2$$

and from Eq. (630), the annual fixed charge against the housing and apparatus is

$$0.01p_3k_{11}d_s^2.$$

The total annual fixed charges, against all factors which make up the cost of a transmission line, and which are a function of the conductor size, are

$$12.31 \times 10^{-5}\, d_s^2L(p_1BW + gp_{Fe}B_{Fe}W_{Fe}) + 0.01p_2LMd_s^2$$
$$+ 0.01p_3k_{11}d_s^2$$

and these charges, according to Kelvin's law, must be equal to the value of the annually wasted energy of the line, which from Eq. (573) is

$$35.64 \times 10^{-6}\rho LA\frac{I^2}{d^2}.$$

Equating the above two statements

$$35.64 \times 10^{-6}\rho LA\frac{I^2}{d^2} = 12.31 \times 10^{-5}d_s^2L(p_1BW + gp_{Fe}B_{Fe}W_{Fe}) + 0.01p_2LMd^2 + 0.01p_3k_{11}d_s^2$$

and from this,

$$I = d_s^2\left(\frac{1}{\rho A}\left[3.45(p_1BW + gp_{Fe}B_{Fe}W_{Fe}) + 280.6p_2M + 280.6p_3\frac{k_{11}}{L}\right]\right)^{\frac{1}{2}}. \qquad (631)$$

Multiplying the left and right members of Eq. (631) by three times the left and right members of Eq (588), respectively,

$$3E_nI = 3d_s^3U\left(\frac{1}{\rho A}\left[3.45(p_1BW + gp_{Fe}B_{Fe}W_{Fe}) + 280\ 6p_2M + 280.6p_3\frac{k_{11}}{L}\right]\right)^{\frac{1}{2}}. \qquad (632)$$

And, from Eq (632),

$$d_s = \left(\frac{\text{r.m.s. volt-amperes per line}}{\left\{\frac{U^2}{\rho A}\left[31.05(p_1BW + gp_{Fe}B_{Fe}W_{Fe} + 2{,}525\left(p_2M + p_3\frac{k_{11}}{L}\right)\right]\right\}^{\frac{1}{2}}}\right)^{\frac{1}{3}} \qquad (633)^*$$

Assuming that, with the proper synchronous reactor equipment connected, the average power factor, along the whole length of the line (for r.m.s. load), is represented by cos θ, Eq. (633) may also be written

$$d_s = \left(\frac{\text{r.m.s. kilowatts per line} \times 1{,}000}{\left\{\frac{U^2\cos\theta}{\rho A}\left[31.05(p_1BW + gp_{Fe}B_{Fe}W_{Fe}) + 2{,}525\left(p_2M + p_3\frac{k_{11}}{L}\right)\right]\right\}^{\frac{1}{2}}}\right)^{\frac{1}{3}}. \qquad (634)$$

* In modified form, this equation was first used by F. K. Kirsten in 1916. The term involving the terminal apparatus was later added by the author.

Equation (634), expressed in general terms and simplified form, is

$$d_s = 10 \times \frac{(\text{r.m s. kilowatts per line})^{\frac{1}{3}}}{[J(F + G + H)]^{\frac{1}{6}}} \tag{635}$$

where

$J = \dfrac{U^2 \cos \theta}{\rho A}$	Factor involving the value of the electrical energy dissipated as heat in the line conductors of one line.
$F = 31\ 05(p_1 BW + g p_{Fe} B_{Fe} W_{Fe})$	Factor involving the cost of the line conductors of one line.
$G = 2{,}525 p_2 M$	Factor involving the cost of that part of the line towers per line, which is a function of the conductor diameter
$H = 2{,}525 p_3 \dfrac{k_{11}}{L}$	Factor involving that part of the cost of high-tension apparatus and housing per line, which is a function of the conductor diameter.

Equation (635) is a mathematical interpretation of Kelvin's law, as modified to include all of the factors involved. The conductor having the diameter d_s, as found from this equation, is the most economical conductor for the conductor material investigated, and the line with which it is used may be called the *line of maximum economy*. An inspection of this equation leads to several very interesting conclusions, among which the following are important:

Conclusions from Economic Relations in Eq. (635).—1. The diameter of the conductor, for the line of maximum economy, is practically independent of the length L of the line, except for relatively short lines. Since, for a given elevation, the voltage to be used is assumed to be proportional to the conductor diameter, it follows that *the voltage used is nearly independent of the length of line.* These conclusions follow since the only factor in the equation which involves the length of line is the factor H. H is inversely proportional to the length of line. Therefore for a very short line the cost of high-tension equipment has an important bearing on the conductor diameter, and the line voltage, whereas, for a very long line this item tends to become a negligible factor. *For very long lines the diameter of conductor and the voltage used are both determined largely by the kilovolt-*

amperes transmitted, and are roughly proportional to the cube root of this quantity.

This conclusion is well illustrated in the curves of Fig. 84. These curves were calculated from the equation for the most economical conductor diameter for certain reasonable assumptions as to cost of towers, and are intended to illustrate relative magnitudes rather than exact values. Copper was assumed to cost 20 cts. per pound delivered, and energy losses were calculated on the basis of 4 mils per kilowatt-hour. It will be seen that for a load of 50,000 r.m.s. kw., the variation of the most economical

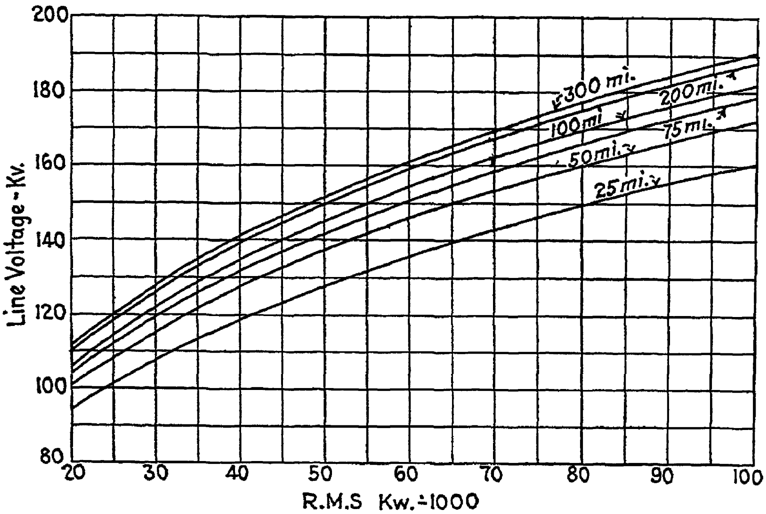

FIG. 84.—Line voltage vs. r m s kilowatts for lines of various lengths.
Type A—copper.
A = 0 004
B = 0 20

voltage is only about 3 per cent for a change in length of line from 100 to 200 miles. For short lines the percentage variation is much greater, since the shorter the line, the more is the cost, and hence the voltage, influenced by the cost of terminal apparatus.

2. Since the line losses are inversely proportional to the square of the conductor diameter, and since, as shown above, for long lines the diameter of conductor is approximately independent of the length of line, it follows from Eq. (635) that *for the line of maximum economy, the losses per unit length of line should be directly proportional to the two-thirds power of the power trans-*

mitted. Or, *for a given amount of power transmitted, neglecting the factor H, the total line losses should vary in direct proportion to the length of the line.* The proof of this statement follows:

$$\text{Loss} = 3I^2R = \text{constant} \times \frac{(kw)^2}{E_n^2 d^2}.$$

But, by equation (588),

$$E_n^2 = \text{constant} \times d_s^2$$

whence

$$\text{Loss} = \text{constant} \times \frac{(kw^2)}{d_s^4}.$$

If in Eq (634) the power transmitted is the only variable, (and this is approximately true in very long lines), then

$$d_s = \text{constant} \times (kw)^{\frac{1}{3}}$$

and

$$\text{Loss} = \text{constant}\ \frac{(kw)^2}{(kw)^{\frac{4}{3}}}$$
$$= \text{constant} \times (kw)^{\frac{2}{3}}.$$

Good engineering will therefore not permit one to assume a given percentage of line loss, independent of the length of line, as a basis upon which to determine other design features. This method of procedure is nevertheless frequently followed. To illustrate further: Given two lines of the same length, one of which transmits twice as much power as the other. For maximum economy, the permissible losses in the two lines should be in the ratio of $1 \div \sqrt[3]{4}$; or the losses in the heavily loaded line should be about 1.59 times as much as the losses on the lightly loaded line.

3. In Eq. (635) the sixth root of the quantity $J(F + G + H)$ appears in the denominator of the expression, whereas only the cube root of the numerator is involved. *It is apparent therefore that considerable errors may be made in estimating either of the factors F, G or H without seriously affecting the accuracy of the result.* The factors F and H can usually be very closely estimated. Considerable latitude in the variation of G is therefore permissible. The factor J, involving the value of the wasted energy, influences the final choice of diameter to a greater degree than either of the factors F, G or H, and should therefore be carefully estimated. For preliminary computations, approximate values of these constants will suffice to arrive at a general

conclusion as to the size of conductor and the line voltage for a given conductor material.

Figure 85 was prepared to illustrate the influence of the value A (selling price of energy) upon the most economical voltage. These curves show that if dependable results are to be had, care must be exercised in evaluating the selling price of energy.

One other rather important conclusion may be drawn from curves such as these. A similar set of curves, drawn for a line of say 220 miles, would show that there is little real likelihood of

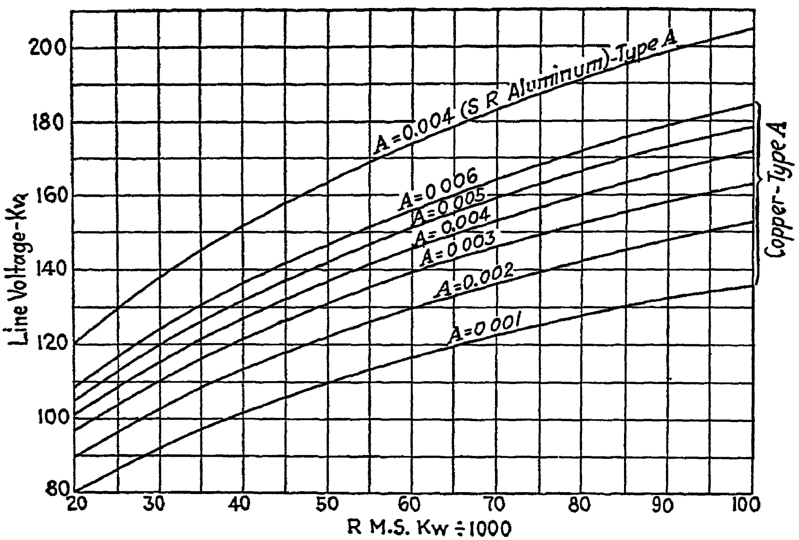

FIG. 85.—Line voltage vs. r m s. kilowats for a 50-mile line and various values of A.

economic conditions in the near future requiring transmission voltages in excess of 220 kv., where stranded conductors of the usual type are used. Figure 84 illustrates the same idea. The topmost curve in Fig. 85 is for steel-reinforced aluminum conductor. It was included to show the marked difference between the economical voltages for the two kinds of conductors.

4. Since the loss of energy in the line is inversely proportional to the square of the line voltage, the theoretically correct conductor to use is the largest one which Eq. (635) will yield. Evidently, then, from consideration of Kelvin's law as expressed by this equation, one is driven to the conclusion that the most economical development of any project requires that all of the

energy which is to be transmitted between a pair of generating and receiving stations, be delivered over a single transmission line, assuming that the power limit of the line is not exceeded

There are certain other conditions, however, such as the necessity for guaranteeing continuity of service, facilitating repair and replacements, etc., which make it practically necessary to construct a minimum of two lines. *If, therefore, it be granted that, for reasons of practical operation, a minimum of two lines is necessary, it seems fair to state that there is no good reason for building more then two lines on any project between a given pair of stations*, provided the voltages called for do not exceed values justified by the then existing state of the art, and provided further that the limits of the capacity of a single line are not exceeded and the question of reliability of service of two lines *vs.* more than two does not enter. It is important to note also that since high voltage and large conductor diameters go together, extra high-potential lines will also be lines of rugged mechanical design, having relatively large factors of safety. The important hazards due to ice and wind loads will be reduced to comparative insignificance in such lines, because they will represent a decreasing percentage of the conductor weight as the conductor diameters increase

Regulation.—The problem of properly controlling the line voltage, for a wide range of load conditions, becomes increasingly complex and costly as the load on the line and the distance of transmission increase. The method of automatic line regulation, now in vogue is the idling of synchronous reactors at the receiver end or at other points of the line. The function of these machines is to adjust the load power factor so that the voltages at both ends of the line may be kept constant for all loads. Automatic voltage regulators, actuated by bus potential coils, operate on the fields of these synchronous motors to change the excitation so that the kilovolt-amperes output of the reactors varies from maximum kilovolt-amperes lagging at no load to maximum leading at maximum line load. The former condition compensates for the condenser characteristics of the line at no load, and the latter compensates for the excessive lagging kilovolt-amperes of the load as it increases to a maximum, the usual load power factor being seldom greater than 0.8 lagging. This system of automatic regulation is the "constant-voltage, variable-power-factor system."

The following discussion will have reference only to the constant-voltage, variable-power-factor system of regulation as applied to lines not exceeding 300 miles in length and carrying blocks of power not in excess of the natural power limit per line.

An exhaustive study was made of required, synchronous condenser capacities for constant voltages at the generating end and the receiving end of lines varying in length from 100 to 300 miles and transmitting blocks of power up to the limit of practical operation. The conductors investigated varied from 250,000 to 2,000,000-cir. mil, metallic cross-section, and the investigation covered three materials, namely, aluminum, copper and steel The results of this investigation may be summarized as follows.

1. The minimum synchronous reactor capacity required is almost proportional to the kilovolt-amperes carried by the line and varies between 40 and 60 per cent of the line kilovolt amperes depending somewhat upon the conductor diameter.

2. The minimum synchronous reactor capacity required decreases somewhat with an increase in conductor diameter for the same load transmitted and for the same length of line.

3. The minimum synchronous reactor capacity required increases somewhat with an increase in the length of line for the same load transmitted and for a given conductor diameter.

4. The minimum synchronous reactor capacity required is practically independent of the conductor material for the same diameter, load transmitted and line length.

From the above and from consideration of Eq. (635) it follows that the cost factors involving line regulation have little or nothing to do with Kelvin's law. An increase in length of line will slightly increase the conductor diameter for maximum line economy. Hence, since the required minimum reactor capacity decreases slightly with increased conductor diameter, and increases slightly with increased line length, an increase of both conductor diameter and line length will not affect the minimum required reactor capacity to any appreciable extent.

The correct method of procedure, in finding the most economical conductor and voltage, is to omit consideration of the required synchronous reactor capacity and proceed with the solution of Eq. (635) without involving costs of automatic regulation requirements.

Coordination of Existing Systems and New Project by Tie Lines. It often happens that the choice of voltage at the receiver end

of a line is strongly influenced by the voltage of existing lines with which the new project is to be coordinated. It is evident that, for long transmission lines, the transmission voltage is much higher than would be suitable for the distribution system which the lines feed. Hence, step-down transformers must be used between the line of the new project and the distribution bus. The same is true of possible tie lines if their operating voltages approach in magnitude that of the new project. The only object gained by equalizing the tie-line voltage with that of the new project is the possibility of interchangeability of transformers and certain other apparatus between the systems in cases of failure. This item, however, is often insignificant as compared with the factors of maximum economy which determine the choice of line voltage, and it should therefore not be given undue weight. In fact, as a first step, each line should be designed independently of any other line which happens to be tied to the same system of distribution in order to obtain maximum overall economy. Should there arise the necessity or desirability of interlinking different transmission systems outside of the distribution system, the use of auto-transformers may in the end be cheaper than the attempt to violate the requirements of Eq (635) for each line, in an effort to equalize the transmission-line voltages for all interlinked projects.

PROBLEMS

1. Derive the empirical Eq (585).

2. A type B, double-circuit line carries 550,000-cir. mil cables of diameter 0.855 in., for which the roughness factor $m_0 = 0.83$ If $\delta = 0.96$ and the line is to operate at 87 per cent of the critical voltage, what is the value of U_B? What is the correct line voltage? Assume that the conductor spacing is given by Eq. (580).

3. To the scale of one unit = 3,000 kw , the ordinates to the average-day load curve for a community are given in the table below.

A. M.		P. M.	
Time	Number of units	Time	Number of units
12	9 2	12 30	13 0
1	8 2	1	15 5
2	6 8	2	15 7
3	6 7	3	15 8
4	6 6	4	16 3
5	6 7	5	18 8
6	9 2	6	19 0
7	13 8	7	15 7
8	16 4	8	14 0
9	16 2	9	13 3
10	15 2	10	12 4
11	15 7	11	11 0
11:55	14 4	11.55	9 4

Calculate the mean kilowatt load and the r.m.s. kilowatt load.

If the average annual load factor of the system is 55 per cent, what is the installed machine capacity?

4. A load equal to that of Problem 3 is to be transmitted over a 125-mile, single-circuit line. The influence of the cost of ground cables may be neglected in calculating the most economical conductor diameter, and the constant G is 25,000. The elevation of the line is 900 ft. Assume. Cost of copper conductor = 18 cts. per pound; $p_1 = p_3 = 10$; selling price of power = \$0.004; weight of copper = 550 lb. per cubic foot; $\delta = 10.5$; cos $\theta = 0.85$; and substations corresponding in cost to type 3 will be used. Find the most economical copper cable diameter and the most economical line voltage.

5. Assume all conditions the same as in Problem 4 except that the steel-reinforced aluminum conductor is to be substituted for copper. On the basis of average percentages of aluminum in the cross-section of the composite cable, we may assume that the effective resistivity of the composite cable is $\rho = 19.1$, and that the weight per cubic foot of composite cable is 210 lb. Find the most economical voltage and conductor diameter for the aluminum line if the composite cable costs 34 cts. per pound.

CHAPTER XIV

VECTOR AND CIRCLE DIAGRAMS OF LINE PERFORMANCE

Vector diagrams for the solution of both short and long transmission lines have appeared from time to time in electrical engineering literature. Those pertaining to short lines usually yield approximate results and cannot therefore be applied with accuracy to lines of great length. Many of those pertaining to long lines are based on approximations, the limits of which must first be carefully investigated in order to satisfy the designer of the degree of accuracy to be expected in a given problem. The series of diagrams[1] given in this chapter are based on the exact equations of the long line developed in Chap. VIII, and contain no approximations. They are extremely useful as an aid in picturing the performance of a line, as a check on the analytical solution of the various quantities involved, and, where a semigraphical solution is desired, as a means of greatly simplifying the work of predicting line performance over any desired range of load. All of the quantities usually desired, such as current, voltage, power, reactive power and power factor at both ends of the line, as well as line regulation, may readily be obtained. Since each diagram presented is a graphical interpretation of the exact line equations, the limit of accuracy is set only by the accuracy of the drawing. Experience shows that accuracy to within 0.5 per cent is generally easily attainable for lines of any length.

Diagrams are presented covering

1. Operation with voltage control by generator excitation.
2. Operation of a straight transmission line by phase control; *i.e.*, the synchronous reactor capacity is all connected to the line at the receiver end, and the line has no intermediate taps.

[1] LOEW, E. A., "Vector Diagrams for Long Lines," *Elec. World*, March 8, 1924

Basic Equations.—Throughout the following discussion the equations upon which the diagrams are based are the equations of current and voltage, namely,

$$E_s = E_rA + I_rB$$
$$I_s = I_rA + E_rC.$$

While, in these equations, the constants A, B and C are used, referring to the line only, it is apparent that, if it should be desirable to introduce the impedances of the raising and lowering transformers and to combine these with the line constants to form new constants of the composite line, this may be done in the manner suggested in Chap. VIII. One may go a step further, and, instead of considering as constant, the terminal voltage at the generator busses, one may assume a constant induced voltage in the generators themselves. The synchronous reactance of the generators would then be added to the reactance of the transformers at the supply end, and their sum would be combined with the line constants to produce the equivalent constants A_0, B_0, C_0 and D_0.

Voltage Control by Generator Excitation.—This method of control assumes constant receiver voltage with supply-voltage variable, as pointed out in the discussion of the preceding chapter.

a. Voltage Diagram.—If the supply-voltage vector be expressed in terms of its two components, its equation is

$$E_s = E_1 + E_2 \tag{636}$$

where

$$E_1 = E_rA \text{ vector volts}$$

and

$$E_2 = I_rB \text{ vector volts.}$$

If the receiver-voltage vector be taken as the zero or reference vector, and since, consistent with the above assumption, its length or size is constant, the component E_1 is a vector of constant length and of fixed angle for all loads. Its length is the constant value

$$E_1 = E_r\sqrt{a_1^2 + a_2^2} \text{ volts} \tag{637}$$

and its angle is the angle of A, that is,

$$_E\theta_1 = \tan^{-1}\frac{a_2}{a_1}. \tag{638}$$

The component E_2 is a variable vector, for it is a function of the variable receiver current I_r; it varies both in length and in angular position. Its length depends upon the amount of the receiver current, and its angle upon the receiver power factor.

The length of E_2 is the product of the lengths of I_r and B, and its angle is the algebraic sum of the receiver power-factor angle and the angle of B Since

$$\begin{aligned} E_2 &= I_rB \\ &= ({}_rI_1 - j{}_rI_2)(b_1 + jb_2) \text{ vector volts} \end{aligned}$$

the size of the vector is

$$E_2 = \sqrt{({}_rI_1^2 + {}_rI_2^2)(b_1^2 + b_2^2)} \tag{639}$$

and its angle is

$$\begin{aligned} {}_E\theta_2 &= \tan^{-1}\frac{b_2}{b_1} - \tan^{-1}\frac{{}_rI_2}{{}_rI_1} \\ &= \tan^{-1}\frac{b_2}{b_1} - \cos^{-1}\text{(receiver power factor).} \end{aligned} \tag{640}$$

These vectors are shown in the diagram of Fig 86. In this figure $E_1 = OM$, $E_2 = MN$, and $E_s = ON$. If the receiver current remains constant, the vector E_2 swings about M as a center with changing receiver power factor. The positions which it would occupy at various power factors are indicated by the radial power-factor lines. On the other hand, if the power factor remains constant and the current only changes, the locus of the vector E_2 is a straight line such as MN. Its angular position remains fixed, but its length changes With constant receiver voltage, the kilovolt-amperes received varies directly as the current, and, by the addition of a suitable kilovolt-ampere scale as indicated, the magnitude and angular positions of all voltages, for any assumed load and power factor, may readily be obtained from the diagram. Figure 86, therefore, yields the generator voltage and the regulation

b. Current Diagram.—The vector diagram of currents results from the current equation for the line in much the same way as does the voltage diagram from the voltage equation. The supply-end current vector consists of the component current vectors

$$I_1 = I_rA \text{ vector amp.} \tag{641}$$

$$I_2 = E_rC \text{ vector amp} \tag{642}$$

Since the receiver voltage is a constant amount, the component vector I_2 is constant in length and fixed in position for

$$C = c_1 + jc_2$$

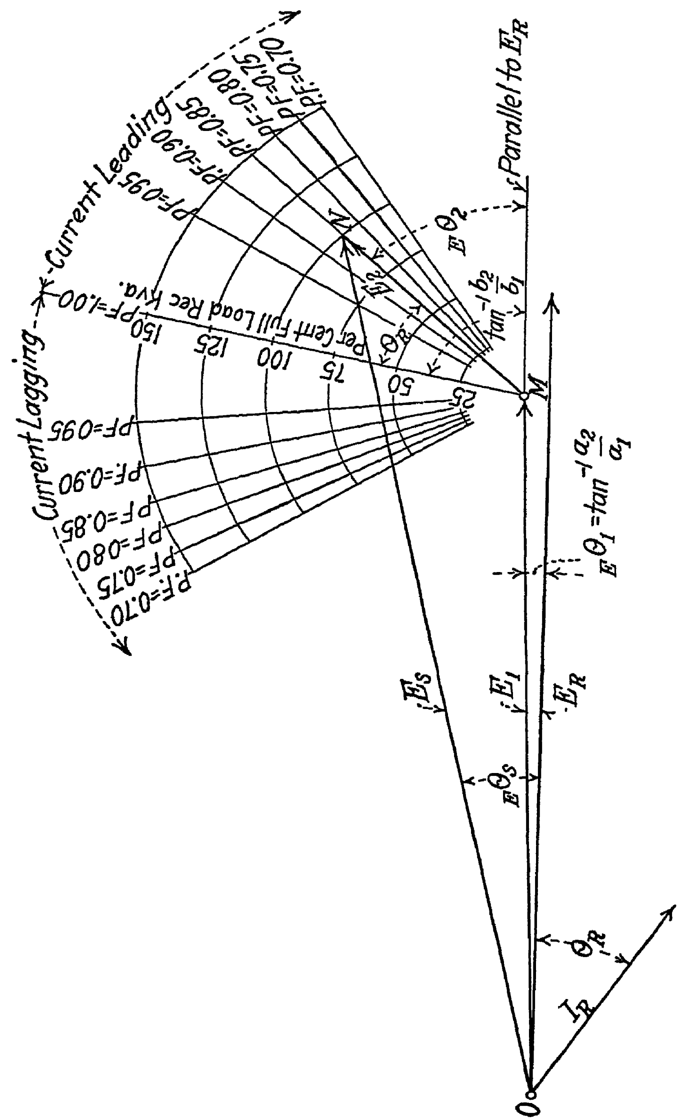

FIG. 86.—Vector diagram of voltages. Control by generator excitation.

and hence the length is

$$I_2 = E_r\sqrt{c_1^2 + c_2^2} \text{ amp.} \tag{643}$$

and its angle is

$$
\begin{aligned}
{}_I\theta_2 &= 0 + \tan^{-1}\frac{c_2}{c_1} \\
&= \tan^{-1}\frac{c_2}{c_1}.
\end{aligned}
\qquad (644)
$$

The component vector I_1 is a vector of varying length and varying angular position, depending upon the receiver load and power factor. At a fixed power factor, it elongates when the load increases and contracts when the load diminishes Under these conditions its locus is a straight line. On the other hand, with constant kilovolt-amperes in the receiver circuit, the length of I_1 remains constant, but it swings through an angle with varying power factor. Its locus is now the arc of a circle.

The length of I_1 is the product of the receiver amperes and the numerical value of the constant A; its angle, with respect to the receiver voltage, is the algebraic sum of the angle of A and the receiver power-factor angle. For, since

$$
\begin{aligned}
I_1 &= I_r A \\
&= ({}_rI_1 - j\,{}_rI_2)(a_1 + ja_2) \text{ vector amp.}
\end{aligned}
$$

its length is

$$I_1 = \sqrt{({}_rI_1^2 + {}_rI_2^2)(a_1^2 + a_2^2)} \text{ amp.} \qquad (645)$$

and its angle is

$$
\begin{aligned}
&= \tan^{-1}\frac{a_2}{a_1} - \tan^{-1}\frac{{}_rI_2}{{}_rI_1} \\
&= \tan^{-1}\frac{a_2}{a_1} - \cos^{-1}\text{(receiver power factor)}.
\end{aligned}
\qquad (646)
$$

After the derived line constants have been computed it is a simple matter to construct the vector diagram of currents from the above equations. From it the supply current may be found graphically for any load and power factor. Such a diagram is shown in Fig. 87.

In this figure, for full load at the receiver and 0.85 power factor, current lagging, the supply current is OQ', having the two components OB and BQ. At the same power factor but for varying receiver loads, the locus of this current is the straight line BQS, while, for full-load kilovolt-amperes and varying power factors, its locus is the arc of a circle, TQU. The angle ${}_I\theta_s$ of the current I_s, referred to the receiver voltage E_r may be measured on the diagram.

Thus, from the current diagram the vector currents at the supply end may be found for any receiver load and power factor From the voltage diagram of Fig. 86, the corresponding angle ${}_E\theta_s$ of the supply voltage is obtained. The difference between these two angles is the generator power-factor angle. From the known supply voltage, current and power factor, the generator output may be calculated. The line loss, equal to the generator output less the receiver input, readily follows.

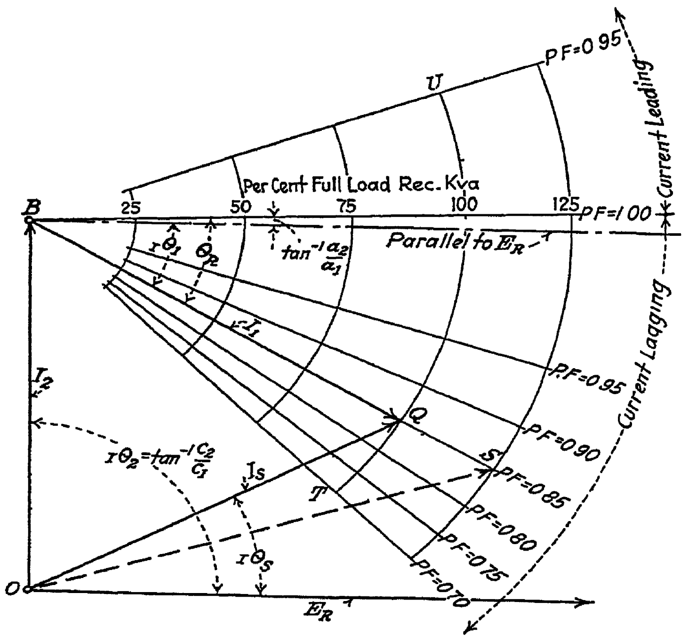

FIG. 87.—Vector diagram of supply currents. Control by generator excitation.

Since the current and voltage diagrams are both referred to the receiver voltage as reference vector, the two diagrams may be superimposed, to good advantage, resulting in a composite diagram upon which the angle between the supply voltages and current may be measured directly. Such a diagram is shown in Fig. 91, for the case of a line operating with phase control.

Voltage Control by Synchronous Reactors at Receiver Only.—By this method of control, constant voltages are maintained at both ends of the line, but both receiving-end and supply-end power factors vary with the loads, as shown in the discussion of the previous chapter. The vector diagrams of current and volt-

age for this case are not as simple as for the case just considered They may be readily drawn on the basis of the theory considered below, however. These diagrams are exceedingly helpful as an aid to the better understanding of line performance under these conditions, besides offering a method by which accurate graphical solutions may be made.

a Receiver-current Diagram.—When a line is operated with synchronous reactors at its receiving end to maintain constant voltages at both ends of the line, the locus of the receiver current is the circle defined by Eq. (461). It is

$$({}_rI_1 + E_rl)^2 + ({}_rI_2 + E_rm)^2 = E_s^2n^2$$

or

$${}_rI_2 = -E_rm + \sqrt{E^2n^2 - ({}_rI_1 + E_rl)^2}.$$

If the X-axis is used as the axis of the active currents ${}_rI_1$ and the Y-axis as the axis of reactive currents ${}_rI_2$, and if lagging currents ($+$ values of ${}_rI_2$) be represented as negative, and leading currents ($-$ values of ${}_rI_2$) as positive in the diagram, the center of the current circle has the coordinates

$$x = -E_rl$$
$$y = E_rm$$

and its radius is

$$R = E_sn.$$

The constants are defined by Eqs. (462), (463) and (464). They are

$$l = \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}$$

$$m = \frac{a_1b_2 - a_2b_1}{b_1^2 + b_2^2}$$

$$n = \frac{1}{\sqrt{b_1^2 + b_2^2}}.$$

Such a diagram of receiver currents is illustrated in Fig. 88, for a particular ratio of $E_s \div E_r$. The receiver current at 100 per cent load is the vector OF; its angle with respect to the receiver voltage is

$$\theta_r = \tan^{-1}\frac{HF}{MH}$$

and the receiver power factor (the power factor of the load plus the synchronous reactors) is

$$\cos \theta_r = \frac{MH}{MF}.$$

b. Receiver-power Diagram.—It has already been explained that the receiver power diagram may be obtained from the current diagram by simply multiplying Eq. (460) through by the receiver voltage. A complete discussion of this diagram is found in Chap. IX.

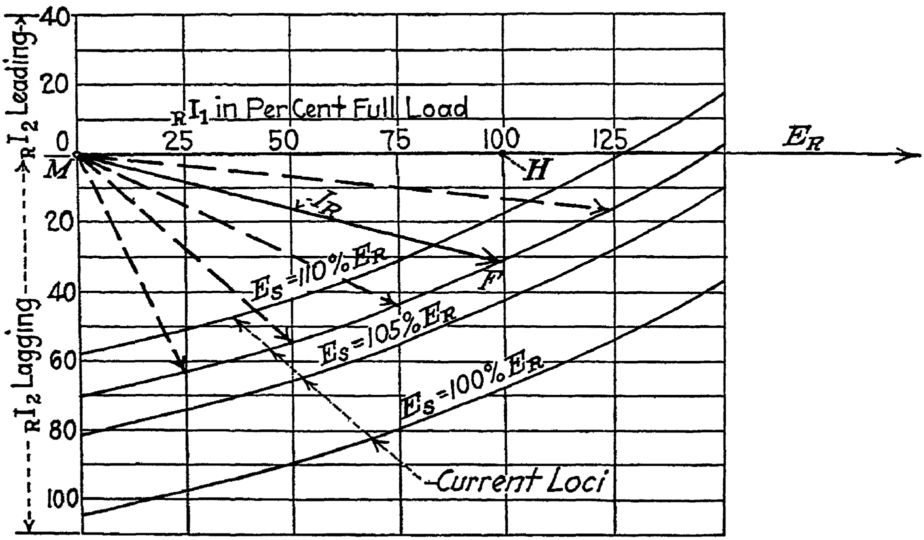

FIG. 88.—Receiver current loci for ratios of $E_s \div E_r$ shown Operation with phase control.

Voltage Control by Synchronous Reactors at Receiving End.—*a. Supply-current Diagram.*—This diagram is obtained from an interpretation of the current equation

$$\begin{aligned} I_s &= I_rA + E_rC \\ &= I_1 + I_2 \text{ vector amp.} \end{aligned}$$

Considering the second component of current first, and using the usual notation

$$\begin{aligned} I_2 &= E_rC \\ &= E_r(c_1 + jc_2) \text{ vector amp.} \end{aligned} \qquad (647)$$

Both E_r and C are constants in this equation. I_2 is therefore a vector of constant length and fixed angle, independent of the load. Its length is

$$I_2 = E_r\sqrt{(c_1^2 + c_2^2)} \tag{648}$$

and its angular position with respect to E_r is

$$_I\theta_2 = \tan^{-1}\frac{c_2}{c_1} \tag{649}$$

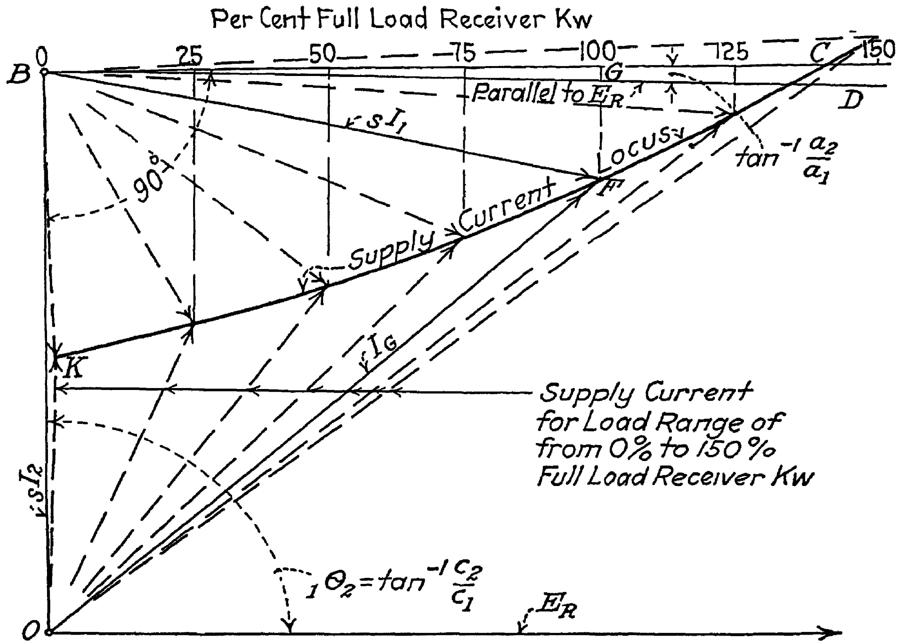

FIG. 89.—Supply current locus Operation with phase control

as shown in Fig. 89. As compared with c_1, c_2 is a large quantity and is positive, while c_1 is always negative. The angle $_I\theta_2$ is therefore an angle slightly over 90° and approaches 90° as a limit as the length of line approaches zero.

The supply current I_s is the sum of the constant vector I_2 above and the variable vector I_1. The latter may be resolved into two components proportional, respectively, to the in-phase and

quadrature components of the receiver load current. Using the complex notation,

$$
\begin{aligned}
I_2 &= I_r A \\
&= ({}_rI_1 - j\,{}_rI_2)(a_1 + ja_2) \\
&= {}_rI_1(a_1 + ja_2) + {}_rI_2(a_2 - ja_1) \\
&= I_{SP} + I_{SQ} \text{ vector amp}
\end{aligned}
\tag{650}
$$

where the vector

$$I_{SP} = {}_rI_1(a_1 + ja_2) \text{ vector amp} \tag{651}$$

has the length

$$I_{SP} = {}_rI_1\sqrt{a_1^2 + a_2^2} \text{ amp.} \tag{652}$$

and the angle with respect to the receiver voltage of

$$\theta_{IP} = +\tan^{-1}\frac{a_2}{a_1}. \tag{653}$$

Similarly,

$$I_{SQ} = {}_rI_2(a_2 - ja_1) \text{ vector amp.} \tag{654}$$

has the length

$$I_{SQ} = {}_rI_2\sqrt{a_1^2 + a_2^2} \text{ amp.} \tag{655}$$

and the angular position with respect to the receiver voltage of

$$\theta_{IQ} = -\tan^{-1}\frac{a_1}{a_2}. \tag{656}$$

From Eqs. (653) and (656) it is apparent that vector I_{SQ} lags vector I_{SP} by 90°.

The locus of vector I_{SP} is a straight line, since its angle $\tan^{-1}\frac{a_2}{a_1}$ is constant for all values of the vector; it is a line $\theta_a = \tan^{-1}\frac{a_2}{a_1}$ degrees ahead of parallelism with E_r. This vector is to be added to I_2. It is therefore drawn from B as an origin, making the angle $\theta_a = \tan^{-1}\frac{a_2}{a_1}$ with the line BD, the latter being parallel to E_r. The full-load value of ${}_rI_1$ is calculated from the known receiver load, or it may be taken from the receiver-current diagram. This value is multiplied by the numeric $A = \sqrt{a_1^2 + a_2^2}$ and is laid off as BG on the line OC. This represents the vector I_{SP} for 100 per cent receiver load. Corresponding vectors for other loads are found by dividing the line BC into sections proportional to the load.

It now remains to add the vector I_{sQ} to the two component vectors OB and BG. It has already been shown that the locus

of the current ${}_rI_2$ is a circle whose center has the coordinates $y = E_r m$, $x = -E_r l$. Since the length of A is constant, it is apparent, from Eq. (650), that the locus of the supply current is also a circle, its center having the coordinates $E_r m\sqrt{a_1^2 + a_2^2}$ and $-E_r l\sqrt{a_1^2 + a_2^2}$ with respect to the axes BC and BK, and its radius being $E_s n\sqrt{a_1^2 + a_2^2}$. With respect to the axes BC and BK, the supply current equation is obtained by multiplying the equation of the receiver current circle through by the constant $\sqrt{a_1^2 + a_2^2}$ This operation yields

$$
\begin{aligned}
I_{SQ} &= -E_r m\sqrt{a_1^2 + a_2^2} \\
&\qquad + \sqrt{E_2 n^2(a \quad + a\ ) - [{}_rI_1(a_1^2 + a_2^2)^{\frac{1}{2}} + E_r l(a_1^2 + a_2^2)^{\frac{1}{2}}]} \\
&= -E_r m_{s1} + \sqrt{E^2 n_{s1}^2 - (I_{SP} + E_r l_{s1})^2} \qquad (657)
\end{aligned}
$$

where

$$
\left.\begin{aligned}
l_{s1} &= l\sqrt{a_1^2 + a_2^2} \\
m_{s1} &= m\sqrt{a_1^2 + a_2^2} \\
n_{s1} &= n\sqrt{a_1^2 + a_2^2}
\end{aligned}\right\} \qquad (658)
$$

Having thus located the center and found the length of the radius, the circle may be drawn. At 100 per cent load the vector I_{SQ} is GF.

It will be observed that, with respect to the axes BK and BC, this diagram is identical in form with the receiver-current diagram already discussed, the only difference lying in the scale used. The supply-current locus may therefore be found from the receiver-current diagram by simply changing the scale of the latter in the ratio of $\sqrt{a_1^2 + a_2^2} \div 1$, and transferring it to the axes BC and BK. By drawing lines perpendicular to BC through the various points representing different percentages of receiver input, and by extending them until they meet the arc KFC, the ends of the corresponding supply-current vectors are found, and thus the vectors themselves are determined, both as to length and slope.

b. Voltage Diagram.—The supply voltage is

$$
\begin{aligned}
E_s &= E_1 + E_2 \\
&= E_r(a_1 + ja_2) + I_r(b_1 + jb_2) \text{ vector volts} \qquad (659)
\end{aligned}
$$

The component E_1 (Fig. 90) is a vector of fixed length and slope. Its length is

$$E_1 = E_r\sqrt{a_1^2 + a_2^2} \qquad (660)$$

and its angular position, referred to E_r is

$$\theta_a = \tan^{-1}\frac{a_2}{a_1}. \tag{661}$$

The second component E_2 is a variable vector, again consisting of two components, one E_{SP}, proportional to the active receiver current and the other E_{SQ}, proportional to the reactive component of the receiver current. That is,

$$\begin{aligned} E_2 &= E_{SP} + E_{SQ} \\ &= (b_1 + jb_2)({}_rI_1 - j{}_rI_2) \\ &= {}_rI_1(b_1 + jb_2) + {}_rI_2(b_2 - jb_1) \text{ vector volts} \end{aligned} \tag{662}$$

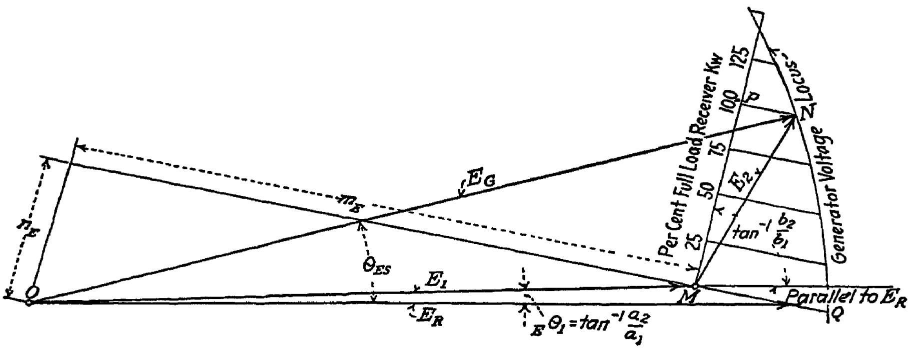

FIG. 90 —Supply voltage locus Operation with phase control

where

$$E_{SP} = {}_rI_1(b_1 + jb_2) \text{ vector volts} \tag{663}$$

and

$$E_{SQ} = {}_rI_2(b_2 - jb_1) \text{ vector volts.} \tag{664}$$

The length of E_{SP} is

$$E_{SP} = {}_rI_1\sqrt{b_1^2 + b_2^2} \text{ volts} \tag{665}$$

and its angular position is

$$\theta_{EP} = \tan^{-1}\frac{b_2}{b_1}. \tag{666}$$

For E_{SQ} the corresponding quantities are
The length is

$$E_{SQ} = {}_rI_2\sqrt{b_1^2 + b_2^2} \text{ volts} \tag{667}$$

and its angle is

$$\theta_{EQ} = -\tan^{-1}\frac{b_1}{b_2}. \tag{668}$$

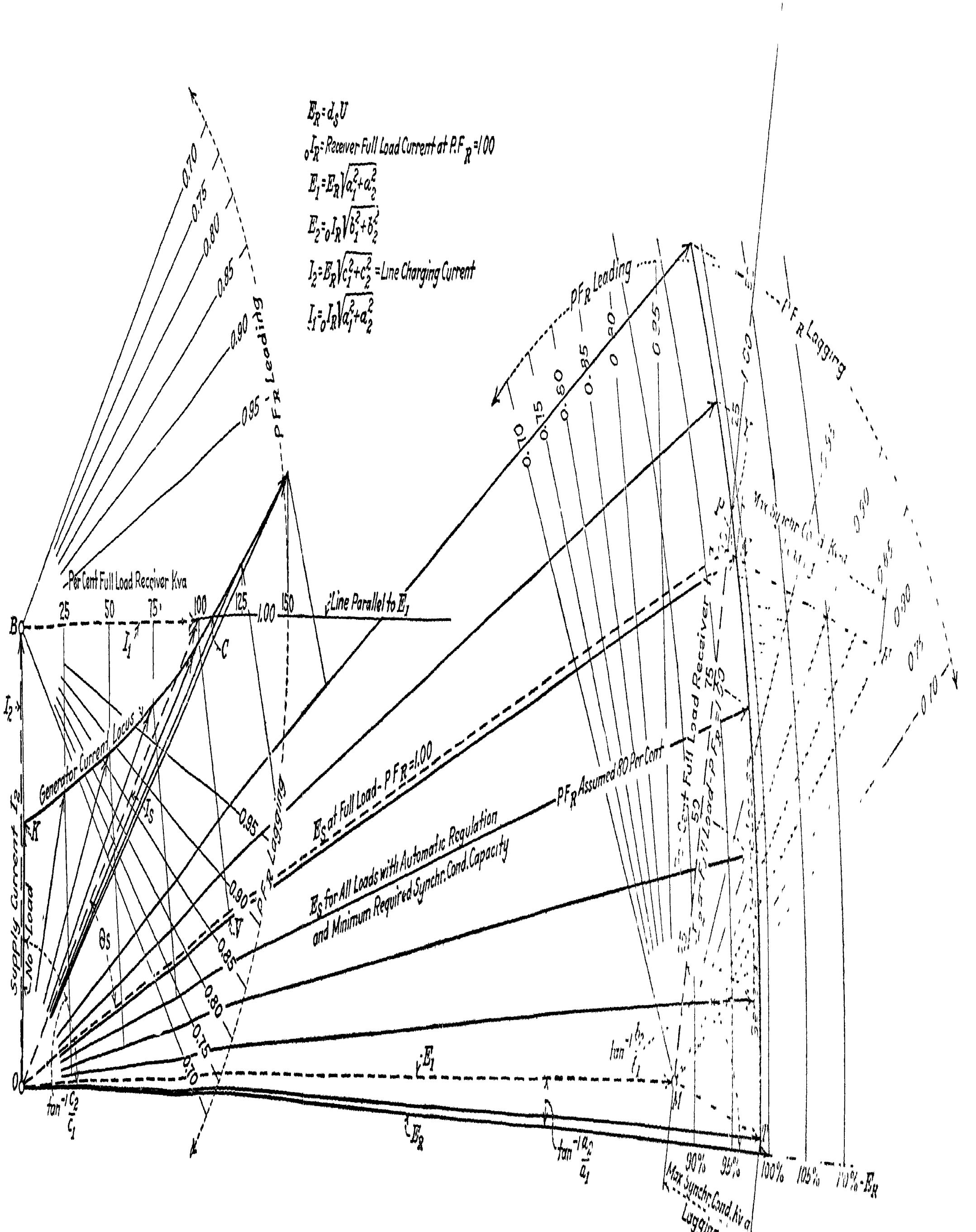

FIG. 91.—Composite diagram of line performance with phase control

a simple matter, once the line constants have been computed and the receiver voltage has been decided upon. The choosing of the proper receiver voltage is a matter of transmission-line economics that has already been considered in detail in Chap. XIII.

For a line having load taps either with or without additional reactor capacity, the problem of line performance may be worked out step-fashion by the method here developed for a line without taps.

Supply Power Circles.—It is desirable to have equations from which one may compute the power at the supply end of a line, or at any other point in the line where the voltage is maintained constant, such as at a tap point in a sectionalized line. For constant supply and receiver voltages the equations of supply power are circles, derived as shown below.

Since the supply power is the dot product of the supply current and voltage vectors, the method consists in finding this product from the fundamental line equations. Thus,

$$\begin{aligned} E_s &= E_1 + jE_2 \\ &= E_r(a_1 + ja_2) + ({}_rI_1 - j{}_rI_2)(b_1 + jb_2) \\ &= E_ra_1 + {}_rI_1b_1 + {}_rI_2b_2 + j(E_ra_2 + {}_rI_1b_2 - {}_rI_2b_1) \end{aligned} \tag{671}$$

and

$$\begin{aligned} I_s &= I_1 + jI_2 \\ &= ({}_rI_1 - j{}_rI_2)(a_1 + ja_2) + E_r(c_1 + jc_2) \\ &= E_rc_1 + {}_rI_1a_1 + {}_rI_2a_2 + j(E_rc_2 + {}_rI_1a_2 - {}_rI_2a_1) \end{aligned} \tag{672}$$

where

$$\left.\begin{aligned} E_1 &= E_ra_1 + {}_rI_1b_1 + {}_rI_2b_2 \\ E_2 &= E_ra_2 + {}_rI_1b_2 - {}_rI_2b_1 \\ I_1 &= E_rc_1 + {}_rI_1a_1 + {}_rI_2a_2 \\ I_2 &= E_rc_2 + {}_rI_1a_2 - {}_rI_2a_1 \end{aligned}\right\} \tag{673}$$

The supply power is

$$P_s = E_s \cdot I_s = E_1I_1 + E_2I_2. \tag{674}$$

Upon carrying out the multiplication indicated by Eq (674), simplifying, and substituting for ${}_rI_1$ and ${}_rI_2$ their respective equivalents $\frac{P_r}{E_r}$ and $\frac{Q_r}{E_r}$, there results

$$P_s = \left(\frac{P_r}{E_r}\right)^2(a_1b_1 + a_2b_2) + P_r(a_1^2 + a_2^2 + b_1c_1 + b_2c_2) + \left(\frac{Q_r}{E_r}\right)^2(a_1b_1 + a_2b_2) + Q_r(b_2c_1 - b_1c_2) + E^2(a_1c_1 + a_2c_2). \tag{675}$$

In order to simplify the notation in Eq. (675), let

$$\left.\begin{aligned} s &= a_1c_1 + a_2c_2 \\ u &= a_1^2 + a_2^2 + b_1c_1 + b_2c_2 \\ v &= bc_1 = b_1c_2 \\ w &= a_1b_1 + a_2b_2 \end{aligned}\right\} \tag{676}$$

Dividing Eq. (675) through by w and using the simplified notation,

$$\frac{P_s}{w} = \frac{P_r^2}{E_r^2} + \frac{P_r u}{w} + \frac{Q_r^2}{E_r^2} + \frac{Q_r v}{w} + \frac{E_r^2 s}{w}. \tag{677}$$

Completing squares

$$\left(\frac{P_r}{E_r} + \frac{E_r u}{2w}\right)^2 + \left(\frac{Q_r}{E_r} + \frac{Ev}{2w}\right)^2 = \frac{P_s}{W} + \left[\frac{u^2 + v^2}{4w^2} - \frac{s}{w}\right]E^2{}_r. \tag{678}$$

Equation (678) is the equation of a family of concentric circles, each representing a constant amount of supply power. The square root of the right-hand member is the radius. Remembering that, in order to represent lagging reactive power in the fourth and leading reactive power in the first quadrant, the sign of the ordinate to the center must be reversed, and making this change to conform to the usual method of representation for leading and lagging currents, the coordinates of the center are:

$$\left.\begin{aligned} \frac{P_r}{E_r} &= -\frac{E_r u}{2w} \\ \frac{Q_r}{E_r} &= +\frac{E_r v}{2w} \end{aligned}\right\} \tag{679}$$

Circles of Constant Power Loss.—By subtracting from the right-hand member of Eq. (678) the power delivered to the receiver circuit, the following expression for the line loss P_L, is obtained.

$$\begin{aligned} P_L &= (P_s - P_r) \\ &= \frac{P_r^2}{E_r^2}w + P_r(u - 1) + \frac{Q_r^2}{E_r^2}w + Q_r v + E_r^2 s \end{aligned} \tag{680}$$

or

$$\left[\frac{P_r}{E_r} + \frac{E_r(u - 1)}{2w}\right]^2 + \left[\frac{Q_r}{E_r} + \frac{E_r v}{2w}\right]^2 = \frac{P_L}{w} + E_r^2\left[\frac{(u - 1)^2 + v^2}{4w^2} - \frac{s}{w}\right] \tag{681}$$

where the symbols u, v, w and s have the meanings assigned to them by (676).

The circles of constant loss, given by Eq. (681), are concentric circles with centers at

$$\left.\begin{aligned} \frac{P_r}{E_r} &= -\frac{E_r(u-1)}{2w} \\ \frac{Q_r}{E_r} &= +\frac{E_r v}{2w} \end{aligned}\right\} \tag{682}$$

and of radii

$$R = \sqrt{\frac{P_L}{w} + E_r^2\left[\frac{(u-1)^2 + v^2}{4w^2} - \frac{s}{w}\right]}. \tag{683}$$

Figure 92 shows the circles of constant power loss for the 200-mile line of Chap. XVI.

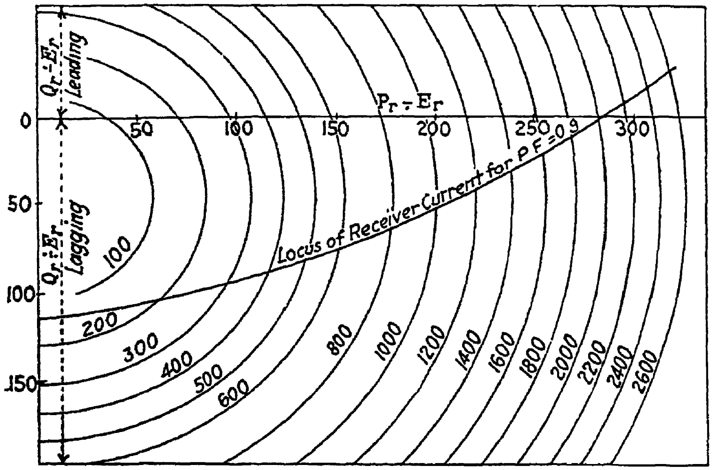

FIG 92.—Loss circles for the 200-mile line of Chapter XVI. Indices on circles indicate kilowatts loss per phase.

Supply Reactive Power Circles.—The reactive power at the supply end of a line, or at any other point in the line at which constant voltage is maintained, such as a tap point in a sectionalized line, may be expressed in terms of the receiver active and reactive powers, the receiver voltage and line constants. The expression is derived from the fundamental equations of current and voltage by a method exactly similar to that used in obtaining the active power equation for the supply end (Eqs (671) and (672)). Since the reactive power is the cross product of current and voltage, the supply end reactive power Q_s is

$$\begin{aligned} Q_s &= E_s \times I_s \\ &= E_1 I_2 - E_2 I_1. \end{aligned} \tag{684}$$

From Eqs (673) and (684),

$$Q_s = (E_r a_1 + {}_rI_1 b_1 + {}_rI_2 b_2)(E_r c_2 + {}_rI_1 a_2 - {}_rI_2 a_1) - (E_r a_2 + {}_rI_1 b_2 - {}_rI_2 b_1)(E_r c_1 + {}_rI_1 a_1 + {}_rI_2 a_2) \quad (685)$$

Upon completing the multiplication indicated in Eq (685), simplifying and substituting for ${}_rI_1$ and ${}_rI_2$ their equivalents $\frac{P_r}{E_r}$ and $\frac{Q_r}{E_r}$, respectively, one obtains the expression

$$Q_s = \left(\frac{P_r}{E_r}\right)^2 (a_2 b_1 - a_1 b_2) - P_r(b_2 c_1 - b_1 c_2) + \left(\frac{Q_r}{E_r}\right)^2 (a_2 b_1 - a_1 b_2) - Q_r(a_1^2 + a_2^2 - b_1 c_1 - b_2 c_2) + E^2(a_1 c_2 - a_2 c_1). \quad (686)$$

In order to simplify the notation in Eq (686), the following substitutions are made:

$$\left.\begin{aligned} s' &= a_1 c_2 - a_2 c_1 \\ u' &= a_1^2 + a_2^2 - b_1 c_1 - b_2 c_2 \\ v &= b_2 c_1 - b_1 c_2 \\ w' &= a_2 b_1 - a_1 b_2 \end{aligned}\right\}. \quad (687)$$

Equation (686) then becomes

$$\frac{Q_s}{w'} = \frac{P_r^2}{E_r^2} - \frac{P_r v}{w'} + \frac{Q_r^2}{E_r^2} - \frac{Q_r u'}{w'} + \frac{E_r^2 s'}{w'}. \quad (688)$$

Completing squares,

$$\left(\frac{P_r}{E_r} - \frac{E_r v}{2w'}\right)^2 + \left(\frac{Q_r}{E_r} - \frac{E_r u'}{2w'}\right)^2 = \frac{Q_s}{w'} + E_r^2 \left[\frac{u'^2 + v^2}{4w'^2} - \frac{s'}{w'}\right]. \quad (689)$$

Here again, Eq. (689) is the equation of a family of concentric circles, each representing a constant value of reactive power Q_s The radius of a circle of the family is

$$R = \sqrt{\frac{Q_s}{w'} + E_r^2\left[\frac{u_2' + v^2}{4w'^2} - \frac{s'}{w'}\right]}. \quad (690)$$

and the center is at

$$\left.\begin{aligned} \frac{P_r}{E_r} &= \frac{E_r v}{2w'} \\ \frac{Q_r}{E_r} &= -\frac{E_r u'}{2w'} \end{aligned}\right\}. \quad (691)$$

The circles of Eq (689) are particularly helpful in finding the synchronous reactor capacity required at some tap point of a sectionalized line in order to maintain constant voltage at that point as well as at the supply and receiver ends of the line.

PROBLEMS

1 A three-phase, 60-cycle line is 250 miles long and delivers 84,000 kw. at full load. The line conductors are three 500,000-cir. mil, copper cables, and the generalized line constants are those found in Appendix D. The receiver voltage is constant and equal to 154 kv. at all loads. The supply voltage is also held constant at all loads by means of synchronous reactors installed at the receiver end of the line. The supply voltage is determined by the condition that its value shall be such as will require the minimum capacity in synchronous reactors at the receiver end. The full-load ratings of the reactors on the leading side are 1 5 times their ratings as generators of lagging kilovolt-amperes. If the *load* power factor may be considered constant,[1] and equal to 87 per cent lagging at all loads, draw a *composite diagram*,[2] and calculate the quantities called for below:

a Draw the receiver-voltage vector diagram showing E_1, E_2 and E_r for full load and 87 per cent power factor.

b Calculate the best ratio of $E_s \div E_r$, and check the calculated value graphically. Locate the corresponding E_s circle on the diagram This will be the supply voltage used.

c. Calculate the synchronous reactor capacity required. Check graphically.

d. Calculate the scale for measuring active and reactive powers on the receiver power diagram, and locate the points representing 0, 25, 50, 75 and 100 per cent loads.

e. Draw the receiver-current vector for each load in (*d*)

f. Locate the center and draw the supply-current circle.

g. Draw the supply-current vectors.
h Calculate the supply power factor.
i. Calculate the supply active power.
j. Calculate the supply reactive power.
k. Calculate the line percentage loss.
} For each load point mentioned in (*d*) above.

l. Draw curves of supply current, supply power factor and percentage line loss against receiver loads in kilowatts, using the latter as abscissas

2. Draw the supply active power circles for the conditions of Problem 1, covering the range of loads from 0 to 100 per cent of full load. Read from

[1] Actually, the power factor at light loads will be considerably less than 87 per cent; but it is the maximum load that determines the synchronous condenser capacity required, and for this load the assumption of a power factor of 87 per cent is reasonable.

[2] Use a voltage scale of 10 kv. to the inch, and remember that graphical methods require that the drawings be carefully made if accurate results are to be expected.

the diagram the supply powers corresponding to each load mentioned in (*d*) of Problem 1, and compare with the values found in (*i*) of Problem 1.

3 Draw the supply reactive power circles for the conditions of Problem 1, covering the range of loads from 0 to 100 per cent of full load. Read from the diagram the reactive powers corresponding to the loads mentioned in (*d*) of Problem 1, and compare with the values found in (*j*) of Problem 1.

4 Draw the loss circles corresponding to the conditions of Problem 1, and covering the range of loads from 0 to 100 per cent of full load. Read from the diagram the loss corresponding to each load mentioned in (*d*) of Problem 1, and compare with the corresponding values computed in (*k*) of Problem 1.

CHAPTER XV

POWER LIMITS OF TRANSMISSION LINES

Power Limits. General.—The question of the power limits of transmission-line networks is a very complex one, and, therefore, one which, in the limited extent of the present volume, must necessarily be treated in a very incomplete manner. Perhaps, however, enough may be said to give a fair idea of the scope of the problem, to indicate what variables are involved, and to suggest how these variables affect the solution.

The power limit of a transmission line is not fixed by the line alone, but depends to some extent upon the nature and characteristics of every piece of apparatus connected to it. It is reached at the point where, for whatever reason, the synchronous apparatus at the two ends of the line break out of step. This may happen for either one of two reasons, namely, (*a*) because, due to *gradually applied loads*, the angle of displacement between the excitation voltages at the two ends of the line has reached its maximum possible value, or, (*b*) because the same angle has been reached due to changes in load resulting from a short circuit, the switching in of load, or other cause. These two conditions under which the limit of output of a line may be reached are designated as, (*a*) *steady-state stability;* and (*b*) *transient stability.* It is apparent that the limit of output which it is possible to reach is greater for (*a*) than for (*b*).

During the past several years the subject has received a great deal of attention at the hands of interested engineers, and considerable literature dealing therewith is now available in the technical journals. A short bibliography of important articles and discussions is given at the end of the chapter. In the discussion following, liberal use has been made of the material contained in these references.

Receiver Power Circle-diagram.—Because this diagram forms the basis of much of the discussion on power limits, it will be given further brief consideration. The equation upon which the diagram is based has been discussed in the chapter on voltage

control. It may be written in any one of several useful forms, one of which is

$$Q_r = E^2\left[-m + \sqrt{\left(\frac{E_s}{E_r}\right)^2 n^2 - \left(\frac{P_r}{E_r^2} + l\right)^2}\right]. \tag{692}$$

If the constants l, m, and n are each multiplied by 1,000, that is, if the "kilo-constants" are used in the equation, then the voltages are in kilovolts, and the active and reactive powers are in kilowatts and kilovolt-amperes, respectively.

The active and reactive powers of Eq. (692) may be represented on the voltage diagram of Fig. 90, here redrawn and slightly modified in Fig 93. The active power is laid off on the axis of reals MP while the reactive power appears on the MQ axis. Again, lagging reactive powers are negative, and leading reactive powers are positive in the diagram.

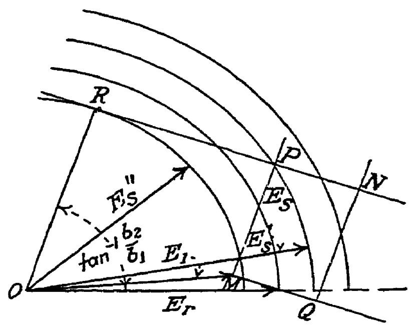

FIG 93 —Theoretical limit of output

That the component voltages E_{MQ} and E_{MP}, of Fig 90, may be converted to power by simply changing the scale of the diagram is clear, for

$$E_{MP} = {}_rI_1\sqrt{b_1^2 + b_2^2}\text{ volts} \tag{693}$$

and

$$E_{MQ} = {}_rI_2\sqrt{b_1^2 + b_2^2}\text{ volts} \tag{694}$$

But

$$P_r = {}_rI_1E_r\text{ watts} \tag{695}$$

and

$$Q_r = {}_rI_2E_r\text{ volt amperes.} \tag{696}$$

Whence, substituting for ${}_rI_1$, and ${}_rI_2$ their values from Eqs. (693) and (694) in Eqs. (695) and (696),

$$P_r = \frac{E_rE_{sP}}{\sqrt{b_1^2 + b_2^2}}\text{ watts} \tag{697}$$

and

$$Q_r = \frac{E_rE_{sQ}}{\sqrt{b_1^2 + b_2^2}}\text{ volt-amp.} \tag{698}$$

Therefore, by multiplying the voltage scale by $\frac{E_r}{\sqrt{b_1^2 + b_2^2}}$, the

distances along MP and MQ become true and reactive powers, respectively.

Power Limit of Line and Transformers in Steady-state Operation. (*a*) *Graphical Method.*—Let us consider the steady-state load limit of the line and transformers, delivering power to a load having a fixed power factor, and let the synchronous condenser capacity available be as large as may be required. The voltage is assumed constant at the low-tension busses at both ends of the line. In other words, under this assumption the impedances of the generators are negligible with respect to the line and transformer impedances. This condition does not exist in practice, yet the case is of interest as illustrating the principle involved as well as preparing the way for later discussion The load is assumed to be added in small increments so that all transient effects are of negligible magnitude.

A diagram similar to Fig. 90 may be drawn to include the influence of transformer impedances by using the equivalent constants in the calculation of l, m and n. In this diagram, Fig. 93, it is seen that for any constant receiver reactive power and constant receiver voltage, the supply voltage is variable, and its locus is a straight line parallel to MP, such as QN, for example. When the receiver power factor is unity, the locus is NP. On the other hand, for any constant receiver load, the locus of the supply voltage is a line parallel to MQ. At the load P it is RN. The minimum supply voltage that will suffice to deliver an assumed load MP, is the voltage $E_s'' = OR$, and if the supply voltage E_s'' is held constant at this value its locus is the arc of the circle shown. MP is the maximum power which can be delivered by the line under assumed conditions. Any further swing of the voltage vector away from the receiver voltage will bring about a reduction of delivered power, and the synchronous apparatus at the two ends of the line will fall out of step. The angle at which this occurs is $\theta_b = \tan^{-1}\frac{b_2}{b_1}$, since OR is parallel to MP. Thus the point of tangency of the normal to NP with the circle of radius E_s'' represents the limit of output of the line and transformers for the particular values chosen for E_s and E_r.

The delivered power could, of course, be further increased by increasing the supply voltage. The effect on the power limit of varying the supply voltage is clearly shown by drawing a series of power circles,[1] each representing a different receiver voltage,

[1] Evans and Sels, "Power Limitations," *Trans.*, A. I. E. E , p. 27, 1924.

the supply voltage remaining the same. Thus keeping the supply voltage constant and assuming various ratios of $E_r \div E_s = s$, such as 1.0, 0 9, 0 8, etc., one may draw a separate power circle for each ratio as in Fig. 94 Since

$$(P_r + E_r^2 l)^2 + (Q_r + E_r^2 m)^2 = E_s^2 E_r^2 n^2 = s^2 E_s^4 n^2$$

the coordinates of the centers are

$$P_r = s^2 E_s^2 l$$
$$Q_r = s^2 E_s^2 m$$

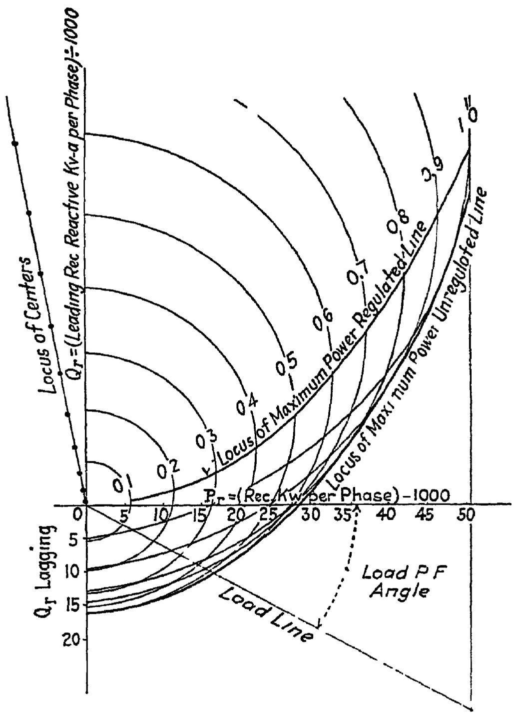

FIG 94 —Line output circles for E_s constant and E_r variable

and the radii are given by the equation

$$R = sE_s^2 n.$$

The locus of the centers is a straight line through the origin making the angle $\theta = \tan^{-1} \frac{m}{-l}$ with the power axis. The distance out on this line from the origin to the several centers is proportional to s^2. A curve through the points of tangency of the power circles with a line parallel to the OQ axis, is the locus of the maxi-

mum steady-state power limit of the regulated line, subject to the original assumptions. The capacity in synchronous condensers, required at this limit for the constant-load power factor represented by the load line, is given by the intercept of a line parallel to the OQ axis and lying between the load line and the point of tangency.

If no synchronous reactors were used at the end of the line to keep constant receiver voltage, the power limit would occur at the intersection of the load line with the envelope to the power circles. Since no power circles extend beyond this limit, the region beyond represents inoperative conditions.

b. Analytical Solution.—Obviously, the above conditions defining the power limits of the line (or of the line including transformers if equivalent constants are used) are mathematically deducible from the power circle equation for the regulated line, one form of which is

$$Q_r = -E_r^2 m + \sqrt{E_s^2 E_r^2 n^2 - (P_r + E_r^2 l)^2}.$$

In general, for all positive values of the radical, Q_r has two values, both representing intersections of the circle with a line parallel to the OQ axis. The lower intersection represents stable operation and the upper one represents unstable operation. For constant values of E_s the quantity Q_r is single-valued only (a) when E_r is zero, an impossible operating condition, or (b) when a line parallel to OQ is tangent to the circle, or at the maximum load limit. The value of P_r which satisfies condition (b) is found by equating the quantity under the radical to zero. For this condition,

$$Q_r = -mE_r^2$$

or, in terms of the ratio $E_r \div E_s = s$ and the constant supply voltage,

$$Q_r = -m\frac{E_s^2}{s^2}$$

$$= -\frac{E_s^2}{s^2}\frac{a_1b_2 - a_2b_1}{b_1^2 + b_2^2} \qquad (699)$$

where the negative sign means leading, reactive kilovolt-amperes.

The condition for corresponding maximum output is

$$P_r + E_r^2 l = E_r E_s n$$

or

$$P_r = E_s^2(sn - s^2 l). \qquad (700)$$

The above value of P_r may still be varied by varying the ratio, s. The ratio which yields the maximum output is found by differentiating with respect to s and solving for maximum power. Thus,

$$\frac{dP_r}{ds} = E_s^2 n - 2sE_s^2 l = 0$$

and

$$s = \frac{E_r}{E_s} = \frac{n}{2l}. \tag{701}$$

In terms of constant supply voltage E_s, the theoretically maximum power which can be transmitted is found by substituting this value of s in Eq. (700), whence,

$$\begin{aligned} {}_{\max}P_r &= \frac{E_s^2 n^2}{4l} \\ &= \frac{E^2}{4(a_1 b_1 + a_2 b_2)}. \end{aligned} \tag{702}$$

Synchronous Motor Supplied from Constant Voltage Mains. Steady-state Stability.—In steady-state stability it is assumed that the load is added in increments so small that the transient conditions associated with the change in load are negligible. If conditions of operation are satisfactory at a given load, and if a slight increment of synchronous motor load is added, all excitations remaining the same, the limit of stability is reached when, under the added increment of load, the excitations are no longer sufficient to supply the power demanded. The assumption of constant excitations is valid, since a drop in voltage must occur before the voltage regulators can operate to bring it back to normal. When the load is the maximum which the given excitation makes it possible to carry, further load increase causes the motor to fall out of step.

A very good way to get a picture of the problem involved in steady-state power limits is to consider the case of a synchronous motor to which energy is being supplied from a constant voltage main. When the motor is unloaded and its excitation is such that the induced voltage E_m is equal to the induced voltage E_r of the equivalent generator representing the bus, then, neglecting losses, the two voltages are in phase. Considering the series circuit through the motor and the equivalent generator, these electromotive forces are in phase opposition as shown in Fig. 95, for E_m in the position 1. If an infinite bus be assumed, the imped-

ance of the equivalent generator is zero and E_r is also the terminal voltage. When the motor is loaded, its excitation remaining constant, its induced voltage drops behind to some new position as 2, for example, in order to permit the passage of the increased current required to carry the additional load. Assuming a constant synchronous impedance, the impedance drop in the motor armature is the potential difference, E_z, and the current through the motor is I_m, making a constant angle with E_z Neglecting

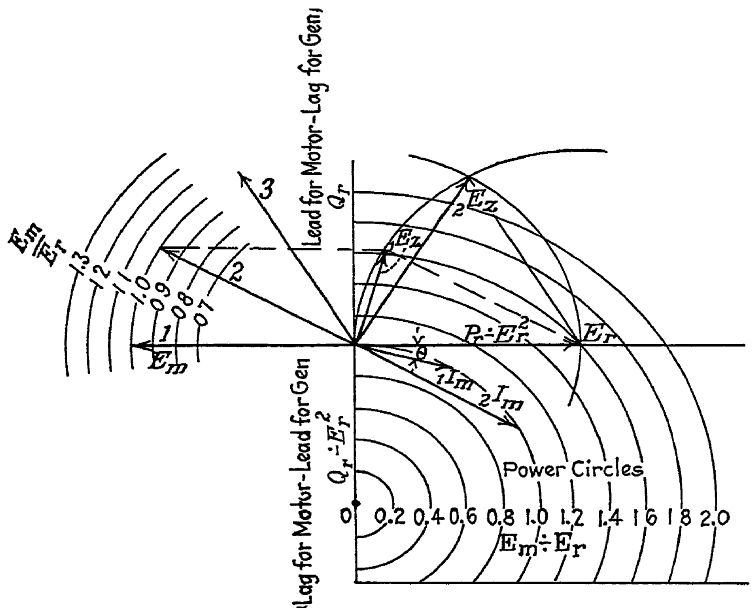

FIG. 95 —Synchronous motor operating on constant voltage bus

motor resistance, this angle is 90°. The power delivered to the motor is

$$P_m = E_r I_m \cos \theta$$

where θ is the angle between E_r and I_m.

When the motor load is indefinitely increased, the vector E_m swings along the arc of a circle, occupying the successive positions 1, 2, 3, etc. The corresponding locus of E_z is likewise the arc of a circle, as is also that of the current I_m.

Since the impressed voltage is constant, the current circles are also circles of power, or of $P_r \div E_r^2$, to different scales. Distances parallel to the axis of $P_r \div E_r^2$ thus represent active power input to the motor, while those parallel to the quadrature axis or axis of $Q_r \div E_r^2$ represent reactive input power to the motor. The

equation of the circles is of the form of Eq. (470) in which $l = 0$ (since the resistance is assumed zero), $m = -\frac{1}{X_m}$ and the radius is $\frac{E_m}{E_r X_m}$, where X_m is the synchronous reactance of the motor. Thus, the equation is

$$\frac{P_r}{E_r^2} + \left(\frac{Q_r}{E_r^2} + m\right)^2 = \frac{E_m^2}{E_r^2 X_m^2}. \tag{703}$$

The circles in the figure indexed 0, 0.2, 0.4, 0.6, etc., are the power circles corresponding to these ratios of voltages $E_m \div E_r$.

As the motor load is increased, further and further, the angle between the voltages E_m and E_r increases accordingly, and with it the power input required to carry the load increases. An angle is finally reached, however, such that any further increase in angle will not result in a corresponding increase in power input to the motor. The resisting torque of the load then exceeds its propelling torque, and the motor falls out of step. In the circle diagram of Fig 95, this condition occurs where a line parallel to the Q_r axis is tangent to the particular power circle representing the existing excitation of the motor. For a motor of zero resistance, the angle at pull-out is 90°. Actually, it is, $\theta = \tan^{-1}\frac{X_m}{r}$.

If, when the motor pulled out, the excitation was such that $E_m \div E_r = 1$, for example, the motor load could be further increased by increasing the excitation to a new value corresponding to some higher value of $E_m \div E_r$, such as 1.2. This increased excitation would increase the load at pull-out to 120 per cent of the former value. Thus it is seen that load conditions requiring a high excitation lead to a high pull-out load, and *vice versa*.

In the above discussion we have considered the impedance of the motor only. In practice the motors are connected in series with additional impedances. These include the impedance of the transmission circuits, and transformers at the supply and receiving ends of the line. Furthermore, the bus is not an "infinite bus," and the impedance of the equivalent generator, representing the generating system, further restricts the maximum load which can be carried under steady-state conditions; for it is the angle between the generator and motor excitation voltages which determines the point of pull-out.

Equivalent System.—When a network, consisting of one or more transmission lines having distributed resistance, inductance

and capacitance, is connected at the supply end through transformers to one or more generating stations in parallel, and at the receiving end through transformers to various kinds of load, the steady-state load limit of the line may still be calculated, provided the complex system can be replaced by an equivalent simple circuit.

The constants of two transmission lines in parallel may be combined to form the constants of an equivalent line. The resistances and the reactances of the transformer banks at the ends of the line may, in turn, be combined with the resultant line constants to form an equivalent line in which the transformer impedances are included. The synchronous impedance of the generating system may also be included, so that, finally, the entire network, including step-up and step-down transformers, the transmission line or lines, and the generators will be represented by four equivalent line constants, A_0, B_0, C_0 and D_0 as pointed out in Chap. VIII.

Equivalent Synchronous Impedance.[1]—In the foregoing, as well as in the following pages containing discussions on power limits, where the synchronous impedances of generators and synchronous condensers are involved, a constant equivalent impedance is used. The synchronous impedance, as ordinarily defined, is a variable quantity, depending upon operating conditions. An equivalent impedance, which may be calculated from the machine characteristics for any given condition of operation, is used instead. This impedance may be considered as a constant for the operating conditions for which it is computed.

Thus, in Fig. 97, are represented the characteristic curves of a synchronous condenser having the usual range of leading to lagging reactive kilovolt-amperes in the ratio of 1.5 to 1. Let X_c' and E_c' be the equivalent synchronous reactance and the equivalent excitation voltage of the synchronous condenser respectively at 100 per cent voltage and given excitation, and let

E_c = actual excitation voltage.
E_t = condenser terminal voltage.
X_c = synchronous impedance at 100 per cent voltage and given excitation as calculated from the condenser V-curves.

[1] Clarke, Edith, "Steady-state Stability in Transmission Systems," paper presented at the Midwinter Convention of the A. I. E. E., New York, Feb. 11, 1926.

From the characteristic curves of Fig 97 and at 100 per cent voltage and for a given excitation such as that corresponding to the curve through P, for example, the slope of the characteristic curve is practically a straight line. Then, since kilovolt-amperes and current are proportional to each other at a given voltage, if the voltage is near normal, the current at any point on the characteristic curve such as P' is

$$I = I_0 + I_x.$$

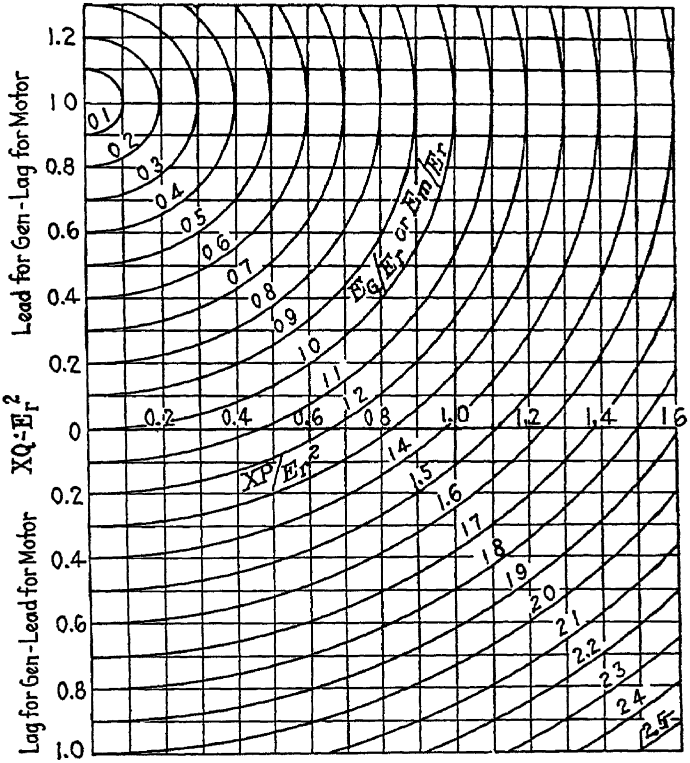

FIG. 96.—Circle diagram for synchronous motor or generator of constant synchronous reactance X, — 100,000 kv -a base.

Expressing terminal voltage in percentage of normal, and reactances in percentage,

$$I_0 = \frac{E_c - 1}{X}$$

and

$$\frac{dE_t}{dI} = \frac{1 - E_t}{I_x} = X', \text{ a constant}$$

whence

$$I = \frac{E_c - 1}{X} + \frac{1 - E}{X'}$$
$$= \left(\frac{E_cX'_c - X'_c + X_c}{X_c} - E_t\right)\frac{1}{X'_c}$$
$$= \frac{E'_c - E_t}{X'_c}.$$

Accordingly,

$$E'_c = \frac{E_cX'_c - X'_c + X_c}{X_c} \qquad (704)$$

and

$$X'_c = \frac{dE_t}{dI_c}. \qquad (705)$$

Calculation of Steady-state Power Limit for the 200-mile Line of Chap. XVI.—To illustrate somewhat in detail how a line operating under known conditions of load and voltage may be

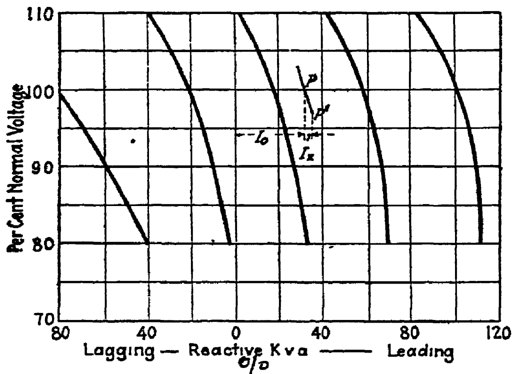

FIG 97.—Synchronous condenser characteristic curves.

investigated for its steady-state power limit, the line whose design is worked out in Chap. XVI will be used as an example. The method[1] to be used is a graphical one. It is based on the power circle diagram, and fits in well with the previous discussion.

a. Given Data.—The following data will be assumed as applying to the 200-mile line to be investigated (Chap. XVI).

[1] For further discussion of this method, see paper entitled, "Steady-state Stability in Transmission Systems," *Proc.*, A. I. E. E., April, 1926, by Miss Edith Clarke. By kind permission of the author Figs. 96, 97 and 98 are reproduced from her paper.

Generators:

Kilovolt-amperes installed = 100,000 or 33,330 per phase
Equivalent synchronous reactance (assumed) 85 per cent.
Frequency = 60 cycles

Length of line = 200 miles

Line constants (Chap. XVI) are

$r = 25.59$ ohms
$x = 162\ 57$ ohms
$b = 10.45 \times 10^{-4}$ mho
$g = 0$
$A = a_1 + ja_2 = 0.9163 + j0\ 0130$
$B = b_1 + jb_2 = 24\ 16 + j158.07$
$C = c_1 + jc_2 = (-0.457 + j101.57)10^{-5}$
$l' = 0.9461$
$m' = 5.652$
$n' = 6\ 254.$

Supply-end transformers:

Kilovolt-amperes installed = 100,000, (33,330 per phase)
Voltages 13,200 − 169,000
Resistance = 1 per cent
Reactance = 10 per cent.

Receiver-end transformers:

Kilovolt-amperes installed = 90,000, (30,000 per phase)
Voltages 150,000 − 13,200
Resistance 1 per cent
Reactance 10 per cent.

b. Equivalent Line Constants—The equivalent line constants, including transformers and generator impedances may be found from the given data by the procedure following. All constants will be referred to the line voltagés, and equivalent single-phase values will be used.

$$I_s = \text{supply current at rated transformer capacity} = \frac{100{,}000}{169} = 591.7 \text{ amp.}$$

$$I_r = \text{receiver current at rated transformer capacity} = \frac{90{,}000}{150} = 600.0 \text{ amp.}$$

X_G = generator synchronous reactance $= \frac{0.85 \times 169{,}000}{591.7} =$ 242.8 ohms.

r_{ST} = resistance of supply-end transformers $= \frac{0\ 01 \times 169{,}000}{591.7}$ = 2 86 ohms.

x_{ST} = reactance of supply-end transformers $= \frac{0\ 10 \times 169{,}000}{591\ 7}$ = 28.56 ohms

r_{RT} = resistance of receiver-end transformers $= \frac{0\ 01 \times 150{,}000}{600}$ = 2.50 ohms.

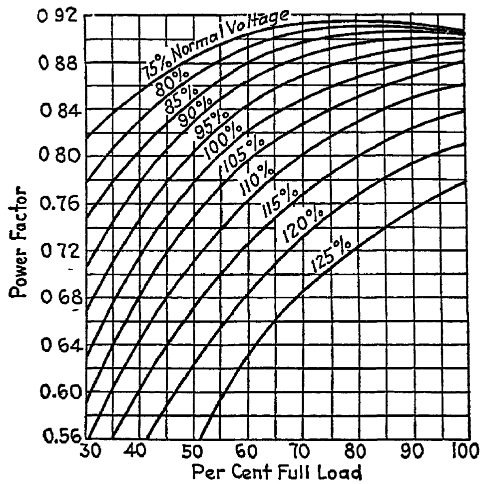

FIG. 98.—Variation of power factor of an average induction motor with load and voltage.

x_{RT} = reactance of receiver-end transformers $= \frac{0.10 \times 150{,}000}{600}$ = 25.00 ohms.

The complex impedances of the various elements to be considered in evaluating the equivalent circuit constants may now be tabulated. They are

Z_G = generator impedance	=	$+ j242.8$
Z_L = line impedance	=	$25.59 + j162.57$
Z_{ST} = supply-end transformer impedance	=	$2.86 + j\ 28.56$
Z_{RT} = receiver-end transformer impedance	=	$2.50 + j\ 25.00$

The equivalent constants for the line, including transformers, by the method of Chap. VIII, are

$$\begin{aligned}
A_0 &= A + CZ_{ST} \\
&= 0.9163 + j0.0130 + 10^{-5}(-0.457 + j101.57)(2.86 + j28\ 56) \\
&= 0.8873 + j0.0158 \\
B_0 &= B + A(Z_{ST} + Z_{RT}) + CZ_{ST}Z_{RT} \\
&= 24.16 + j158.07 + (0.9163 + j0.0130)(5\ 36 + j53.56) \\
&\quad + 10^{-5}(0.457 + j101.57)(-706.86 + j142.90) \\
&= 28.23 + j206\ 50 \\
C_0 &= C = 10^{-5}(-0.457 + j101.57) \\
D_0 &= A + CZ_{RT} \\
&= 0\ 9163 + j0.0130 + 10^{-5}(-0.457 + j101\ 57)(2\ 50 + j25\ 00) \\
&= 0.8909 + j0\ 0154.
\end{aligned}$$

The equivalent constants of the line, including transformers and the synchronous impedance of the generators, are

$$\begin{aligned}
A_{00} &= A_0 + CZ_G \\
&= 0.08873 + j0.0158 + j242.8 \times 10^{-5}(-0.457 + j101.57) \\
&= 0\ 6407 + j0.0147 \\
B_{00} &= D_0Z_G + B_0 \\
&= j242.8(0.8909 + j0.0154) + 28.23 + j206.50 \\
&= 24.49 + j422\ 81 \\
C_{00} &= C_0 = 10^{-5}(-0\ 457 + j101.57) \\
D_{00} &= A_0 = 0.8873 + j0\ 0158.
\end{aligned}$$

For line and transformers alone, the values l'_0, m'_0 and n'_0 (for $E_s \div E_r = 1$) are

$$l_0' = \frac{1{,}000(0.9163 \times 28.23 + 0.0158 \times 206\ 50)}{\overline{28.23}^2 + \overline{206.50}^2} = 0.67$$

$$m_0' = \frac{1{,}000(0.9163 \times 206.50 - 0\ 0158 \times 28.23)}{\overline{28.23}^2 + \overline{206.50}^2} = 4.35$$

$$n_0' = \frac{1{,}000}{(\overline{28.23}^2 + \overline{206.50}^2)^{\frac{1}{2}}} = 4.80.$$

Similarly, for line, transformers and generators, the equivalent power circle constants are

$$l'_{00} = 0.12$$
$$m'_{00} = 1.51$$
$$n'_{00} = 2.36, \text{ (for } E_1 \div E_r = 1)$$

where E_1 is the excitation voltage of the generator.

In this case the line is designed for a constant supply voltage of 169 kv. and a constant receiver voltage of 150 kv. Hence $E_s \div E_r = 1.127$ and, for operating conditions,

$$n'_0 = 1.127 \times 4.80 = 5.41.$$

The line of Chap. XVI was designed for a full-load of 80,000 kw. approximately. Let the load be assumed to consist of the following:

1. P_m, a synchronous motor load of 25,000 kva. The motors are 80 per cent loaded, have 100 per cent synchronous impedance, and operate at unity power factor.

2. P_I induction motor load of 30,000 kw., operating at 80 per cent power factor under normal voltage.

3. P_R resistance load of 30,000 kw.

4. A synchronous condenser of 45,000 kva. capacity is connected to the line at the receiver end (Chap XVI).

For normal operating conditions of the line, $E_r = 150$ kv. and $E_s = 169$ kv. It is required to investigate the line for steady-state stability under the given load.

Using the constants $l'_0 = 0.67$, $m'_0 = 4.35$, and $n'_0 = 5.41$, the circle 1 of Fig. 99 is drawn. This is the power circle for line and transformers, and represents the steady-state operating conditions, assuming fixed voltages of 150 and 169 kv. at the two ends of the line. Using the constant $l'_{00} = 0.12$, $m'_{00} = 1.51$ and $n'_{00} = 2.36$, the power circle is drawn for line, transformers and generator numbered 1.0 of family 2. The other circles of the family may be drawn by using the same center, but various radii bearing the ratios 1.1, 1.2, 1.3, etc., to the radius $n'_{00} = 2.36$. These are the power circles of line, transformers and generators. They represent the relation between active and reactive powers for the ratios given, the ratio being the quotient of generator excitation voltage and receiver voltage, referred to the high side. Each circle represents a particular excitation voltage.

Given the power circles of Fig. 99, the conditions of receiver loading and the characteristic curves of the various kinds of apparatus comprising the receiver equipment, the circuit may be tested for steady-state stability. As already pointed out, steady-state stability assumes additions of load to be made in increments so small that the transients associated with the change are negligible.

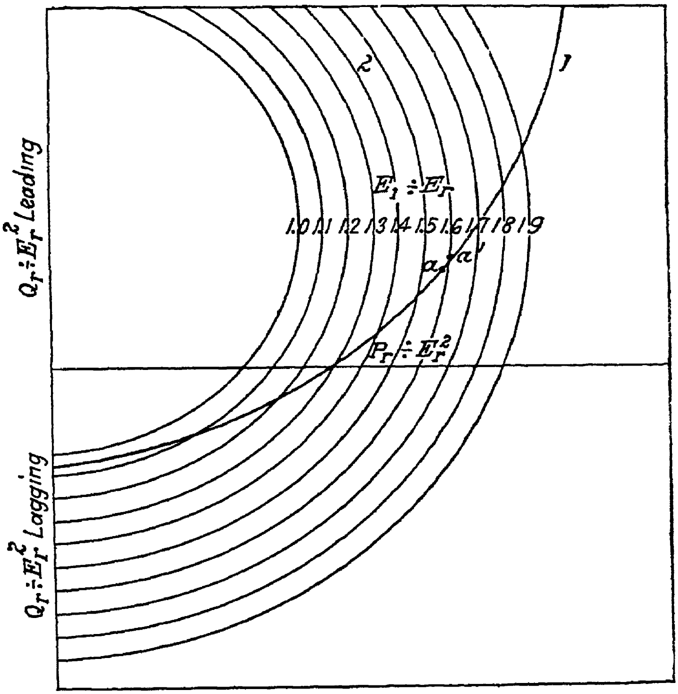

FIG 99 —Power circles of line for, (1) line and transformers, (2) line, transformers and generators

To test for stability at a given load the procedure is as follows: Calculate $P_r \div E_r^2$ for the assumed load and the normal voltage, and locate on curve 1 the point corresponding thereto. The particular curve of the family 2 that passes through this point determines the ratio of $E_1 \div E_r$ which is required, and the corresponding value of E_1 may be computed. Next assume that the receiver voltage has dropped slightly, say to 98 per cent of its original value, but that all excitations have remained unchanged. Calculate the active and reactive powers for this new voltage,

and, likewise, the values of $P_r \div E_r^2$ and $Q_r \div E_r^2$. Locate on the circle diagram the points corresponding to the new values of $P_r \div E_r^2$ and $Q_r \div E_r^2$, and note the value of $E_1 \div E_r$ corresponding to it. Compute the new value of E_1 from this ratio. If it is less than the original excitation voltage calculated for normal receiver voltage, the system is stable; if greater, it is unstable. The limit of stability can only be determined by the method of trial and error, that is, by assuming various loads and testing for each. When that load is found for which the excitation voltages at the reduced voltage and at normal voltage are just equal, the limit of stable operation has been found.

At normal voltage,

$$E_r = 150 \text{ kv.} = 100 \text{ per cent receiver voltage.}$$
$$P_r = 80{,}000 \text{ kw. total load over line.}$$
$$\frac{P_r}{E_r^2} = 80{,}000 \div 22{,}500$$
$$= 3.56 \text{ (locates point } a \text{ on circle 1).}$$
$$\frac{Q_r}{E_r^2} = 0.99 \text{ corresponding to point } a \text{ on diagram.}$$
$$+jQ_r = j0\ 99 \times 22{,}500$$
$$= +j22{,}300 \text{ reactive kva. over line.}$$
$$\frac{E_1}{E_r} = 1.57 \text{ from circles (2) at point } a.$$
$$E_1 = 1.57 \times 150$$
$$= 235.5 \text{ kv. excitation voltage at generator end.}$$
$$P_m + jQ_m = 200{,}000 + j0 = \text{synchronous motor load}$$
$$P_I - jQ_I = 30{,}000 - j22{,}500 = \text{induction motor load.}$$
$$P_R + jQ_R = 30{,}000 + j0 = \text{resistance load.}$$

Adding,

$$80{,}000 - j22{,}500 = \text{total motor and resistance load.}$$
$$P_r + jQ_r = 80{,}000 + j22{,}300 = \text{total receiver load over line.}$$

$Q_c = +j22{,}300 - (-j22{,}500) = +j44{,}800$ kva. (synchronous condenser).

X_m = synchronous motor impedance in per cent on 100,000 kva. base,

$$= \frac{100{,}000 \times 100}{25{,}000}$$
$$= 400 \text{ per cent.}$$
$$\frac{X_m(P_m + jQ_m)}{E_r^2} = \frac{400(20 + j0)}{100^2}$$
$$= 0.8 + j0.0.$$

Locating these values of Q_m and P_m on the circle diagrams for the synchronous motor (Fig. 96), $E_m \div E_r$ is found to be

$$\begin{aligned}\frac{E_m}{E_r} &= 1.285 \\ E_m &= 1.285 \times 150 \\ &= 193 \text{ kv.} = \text{motor excitation voltage.}\end{aligned}$$

When the receiver voltage drops to 98 per cent of its normal value,

$$\begin{aligned}E_r &= 0.98 \times 150 \\ &= 147 \text{ kv.}\end{aligned}$$

P_m = 20,000 kw. (synchronous motor load remains constant).

P_I = 30,000 kw. (induction motor load remains approximately constant).

P_R = 28,800 kw. (resistance load varies as E_r^2).

Adding,

$$\begin{aligned}P_r &= 78{,}800 \text{ kw. total receiver load.} \\ \frac{P_r}{E^2} &= \frac{78{,}800}{147^2} \\ &= 3.64.\end{aligned}$$

E_m = 193 as before, since the excitation is assumed constant.

$$\left.\begin{aligned}\frac{E_m}{E_r} &= \frac{193}{147} \\ &= 1.31 \\ \frac{X_m P_m}{E_r^2} &= \frac{400 \times 20}{98^2} \\ &= 0.834\end{aligned}\right\}$$

Using the bracketed values in connection with the circle diagram for the synchronous motor (Fig. 96), we find

$$\frac{X_m Q_m}{E_r^2} = 0.01$$

and from the characteristic curves of the synchronous condensers (Fig. 97), it is seen that the leading reactive kilovolt-amperes has increased about 3 per cent, or

$$
\begin{aligned}
Q_c &= +j46{,}400 \\
Q_m &= +j\frac{E_r^2}{x_m} \times 0.01 \\
&= \frac{(0.96 \times 0.01) \times 10^5}{4.00} \\
&= j240 \text{ kva.}
\end{aligned}
$$

The power factor on the induction motors has increased to about 81 per cent (see Fig. 98 for induction motor characteristics) and hence, under the reduced voltage,

$$Q_I = -j21{,}700.$$

Adding,

$$
\begin{aligned}
Q_r &= +j24{,}940 \text{ kva.} \\
\frac{Q_r}{E_r^2} &= \frac{24{,}940}{147^2} \\
&= 1.15.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{P_r}{E_r^2} + j\frac{Q_r}{E_r^2} &= 3.64 + j1\ 15, \text{ located at } a' \text{ (Fig. 99).} \\
\frac{E_1}{E_r} &= 1.60 \\
E_1 &= 1.60 \times 147 \\
&= 235.2 \text{ kv.}
\end{aligned}
$$

Since the required excitation voltage is less than the original 235.5 kv., the system is stable at 80,000-kw. load. A similar test, made with a load of 85,000 kw., shows the line to be unstable at that load. Hence the limit of steady-state ability for the line under assumed conditions of load lies between 80,000 and 85,000 kw.

Transient Stability.—Thus far the discussion has dealt almost entirely with the power limits of lines under steady-state operation. As a matter of fact, the limits of output are more likely to be reached on account of transient conditions. Especially, when a line is already operating under heavy load, a sudden application of additional load, due to switching, short circuit or other condition of operation, may set up transient conditions which will

cause the synchronous load to fall out of step with the generators, even though, if gradually applied, the added load could have been carried without reaching the load limit of the line.

When a load is suddenly applied the impedance of the receiver circuit is suddenly reduced and a sudden increase of generator current results. Due initially to the leakage reactance of the generator and later to its synchronous reactance, the increased current causes a drop in terminal voltage, which, in turn, permits the regulators to function, tending to bring the voltage back to normal. While the above series of events take place, the generator and motor armatures are swinging apart in phase; that is, the angle between their excitation voltages is increasing in order to permit of the increased interchange of power demanded by the new situation. This demands that the rotating masses momentarily slow down and later again accelerate when the correct new phase position has been reached. During the slowing down interval, some of the increased energy demanded is supplied from the energy of rotation of the generator rotor. Another element which assists in supplying the increased demand is the fact that the suddenly increasing armature current induces additional current in the field circuit, thereby momentarily increasing the excitation voltage. The readjustment of the rotor positions and the dropping of the terminal voltage at the generators go on simultaneously. The farther the voltage drops, the greater is the required phase displacement between the motor and generator excitation voltages. It is thus a contest between the exciters and regulators on the one hand, and the decelerating rotors on the other. If, during the initial swing, the angle between the motor and generator excitation voltages reaches the critical value before the regulators and exciters can operate to raise the voltage again, and thus limit the angle to a lesser value, the system will fall out of step. The inertia of the negatively accelerating rotors tend to cause an over swing, thus further endangering the stability.

Methods of Increasing Output Limits of Lines.—The steady-state power limits may obviously be increased by any measures which will reduce the impedance of the circuit involved. These include the use of the so-called "split conductor" suggested by Percy Thomas, as a means of reducing line reactances; two or more parallel transmission lines instead of a single line; transformers having lower reactances; generators designed for lower reactances; lower frequencies; shunt reactors at the generators,

thus requiring higher excitations; intermediate synchronous condenser stations in long lines.

Transient stability conditions are improved by using exciters whose voltages respond rapidly to changes in field resistance, and at the same time using quick acting voltage regulators to control the exciters.

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CHAPTER XVI

EXAMPLE OF LINE DESIGN AND PERFORMANCE CALCULATIONS

In the preceding chapters the theory underlying transmission-line design, as well as line performance, has been considered. This, the last chapter, will be devoted to the application of this theory to the design of a hypothetical problem, by way of illustration. In this problem, as with any problem, certain assumptions must be made as to the conditions prevailing. These assumptions will largely determine the design which follows. For example, an examination of Eq. (635), and a reading of the discussion following, shows that the conductor diameter and the voltage of a proposed line are chiefly dependent upon the r.m s. load to be transmitted. Accordingly, assumptions affecting the amount of installed generating equipment, available power and load factor greatly influence the subsequent design. The proper selection of these factors often depends upon circumstances which can scarcely be discussed to advantage here. In choosing an illustrative example, therefore, a simple case is chosen in which the load transmitted is assumed to be determined entirely by the load curve of the distribution center.

TABLE 26 —ASSUMED DATA

Location of project	Eastern Washington
Number of lines..	One
Length of line	200 miles
Continuously available power at generator terminals (average demand)	50,000 kw.
Power factor at peak load	90 per cent
Average annual load factor	55 per cent
Frequency	60 cycles per second
Maximum clearance of conductor to ground	30 ft.
Maximum elevation of line above sea level	2,000 ft.
Conductor materials considered	Copper and aluminum

The transmission voltage chosen for the problem is the one which fits the conductor diameter demanded by Eq. (635). It is realized, of course, that the final choice of voltage may be

affected by pre-existing standards, by the necessity for linking the new line with other systems, and other considerations. These will not be considered here.

Root-mean-square Kilowatts per Line.—The curve of Fig. 100, plotted to any convenient scale, is assumed to represent the average-day load curve of the community served. The curve is obtained by averaging a sufficient number of day load curves at regular intervals throughout the year, to secure a fair average for the community. This curve was obtained by averaging the load curves for every Tuesday of the year.

The mean load of the average day is readily found by integrating the area under the average-day load curve in terms of the

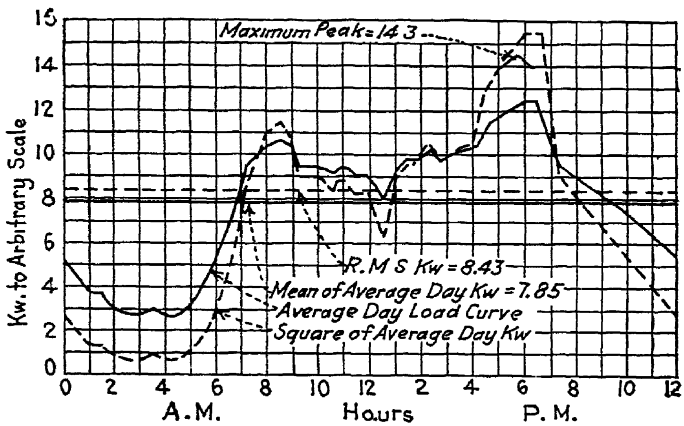

Fig. 100.—Average-day load curve

arbitrary unit chosen and dividing this area by 24, or by scaling the area under the curve with the planimeter set to read the square inches, and dividing by the length of the base in inches. The result must again be translated into the units of the arbitrary scale. The ordinates of the horizontal full lines in the figure indicate the mean of the average-day load. As a check, the area under the horizontal full lines should equal the area under the average-day load curve.

The dotted curves are obtained by squaring the ordinates of the average-day load curve. The mean ordinate of the dotted curve is also found by integrating the area under this curve and dividing the result by the length of the base. The r.m.s. value is the square root of the mean square and is shown by the horizontal broken line.

The half-hour average annual load factor is defined as the ratio of the average to the half-hour peak demand of the year. This load factor may vary from 40 to 65 per cent or more. Here it is assumed to be 55 per cent. The storage provided by the basin behind the dam will be at least sufficient to take care of the variation in demand from the average, during any time of day or year. Thus, with 50,000 kw. of power continuously available at the generator terminals, if the power factor at peak load is 90 per cent, the installed machine capacity required is

$$\begin{aligned}\text{Installed capacity} &= \frac{50{,}000}{0.55 \times 0.90} \\ &= 101{,}000 \text{ kva., say } 100{,}000 \text{ kva.}\end{aligned}$$

To the scale of the curve the average kilowatt is 7 85, while the r m s value is 8 43. Therefore, the r.m s load to be supplied to the line is

$$\begin{aligned}\text{R.m.s. kilowatts supplied} &= \frac{50{,}000 \times 8\ 43}{7.85} \\ &= 53{,}700 \text{ kw}\end{aligned}$$

Factor J. Value of Wasted Energy.—From Eq (635) the factor pertaining to the value of the energy wasted as heat in the line conductors is

$$J = \frac{U^2 \cos \theta}{\rho A}.$$

Since only one line is contemplated, a single circuit tower of type A is the only type considered. The maximum altitude of any point on the line is assumed to be 2,000 ft. For this elevation the altitude factor is $\delta = 0.92$, from Table 24. The roughness factor for stranded cable may be assumed as $m_0 = 0.82$. Then, if we design the line so that, at the point of maximum elevation, the actual line voltage will be about 10 per cent under the critical disruptive value, the constant γ of Eq. (587) is 0.90. Thus,

$$\begin{aligned}m_0\delta\gamma &= 0.82 \times 0.92 \times 0.90 \\ &= 0.678\end{aligned}$$

and, by interpolation from the values of Table 23,

$$U_A = 116{,}900.$$

The resistance, per mil-foot of conductor and for copper and aluminum, may be estimated for any assumed average operating temperature from the equations

$$\rho_{Al} = 17.01[1 + 0.0039(t - 20)] \text{ for aluminum}$$
$$\rho_{Cu} = 10.37[1 + 0.00382(t - 20)] \text{ for copper}$$

where

t = assumed average operating temperature of line conductors in degrees centigrade.

The mean temperature for the year and locality may be obtained from the records of the U. S. Weather Bureau. For the project here considered, it is 48° F. The line conductors, in order to radiate the heat developed in them, must operate at a temperature somewhat higher than that of the surrounding air. If this difference is taken as an average of 15° F., the average temperature of the line conductors is 63° F , or 35° C., and the resistivities of aluminum and copper under average operating conditions are

$$\rho_{Al} = 17.01[1 + 0\ 0039(35\text{–}20)] = 17\ 1 \text{ ohms}$$
$$\rho_{Cu} = 10.37[1 + 0.00382(25\text{–}20)] = 10\ 4 \text{ ohms.}$$

The weighted average selling value of a kilowatt-hour of energy, represented by the constant A, is difficult to estimate The value of a kilowatt-hour will depend upon the locality of the plant, market conditions, the distance of transmission, the time of day and year (if generated from water power with incomplete storage), whether or not steam generation is required at peak loads and other factors. A study should be made of the revenues derived from several years of past operation together with a forecast of future revenues with the new project added to the existing system. The value of A may vary from as low as 0.002 for some western hydroelectric plant which was constructed at a very low unit cost, where the transmission line is short, and where an ample supply of power exists, to perhaps 0.007 where conditions of generation and transmission are less favorable. For the purpose of the problem to be considered here, 0 006 will be assumed. The power factor at maximum load is assumed to be 0.90.

The factor J and the constants used in deriving it are summarized in Table 27.

TABLE 27

Item	Symbol	Value for aluminum	Value for copper
Constant U of equation .	U	116,900	116,900
Constant U^2..	U^2	136.7×10^8	136.7×10^8
Resistivity at 20° C	ρ_{20}	17.01	10.37
Selling price per kilowatt-hour (dollars)	A	0.006	0.006
Power factor of r m s. load (assumed).	$\cos \theta$	0.90	0.90
$J = U^2 \cos \theta - \rho A$ (Eq (64))	J	12.0×10^{10}	19.8×10^{10}

Factor F. Annual Fixed Charges Due to Line Conductors.—By Eq. (634) the annual fixed charges due to the line conductors and ground cables of one line are

$$31.05(p_1BW + gp_{Fe}B_{Fe}W_{Fe}).$$

The several factors involved in the constant F are listed in Table 27 for the two kinds of conductor materials.

The values of p_1 and p_{Fe}, representing interest and depreciation on the conductors and ground cables, will vary somewhat for different projects, depending upon the condition of the money market and the credit of the company undertaking the construction. It is a matter of judgment how much should be allowed for the depreciation of line conductors. The depreciation of copper conductors is relatively little The values allowed in this problem are illustrative rather than exact.

TABLE 28

Item	Symbol	Aluminum	Copper
Per cent interest and depreciation on line conductor	p_1	10	9
Cost of conductor in dollars per pound	B	0.38	0.19
Weight per cubic foot of conductor material.	W	168.5	555
Guard cable constant (Eq. 625) .. .	g	0.167	0.333
Per cent interest and depreciation on guard cables	p_{Fe}	12	12
Cost of guard cables in dollars per pound	B	0.12	0.12
Weight per cubic foot of guard cable material	W_{Fe}	490	490
Constant F of Eq. (64) .	F	23,540	36,760

The costs, per pound of stranded conductor and ground wires in the sizes and amounts required for the project, are represented by B and B_{Fe}. The values given in the table are approximate prices for these materials at the time and place of writing.

The values of W and W_{Fe} are the weights per cubic foot of the materials composing the line conductors and ground cables, respectively, corresponding to the densities of the materials, as given in Chap. II. The constants F, together with the quantities used in their calculation, are shown in Table 28.

Factor G, Tower Cost.—By Eq (635) that part of the cost of transmission line towers, which is a function of conductor diameter, is the constant

$$G = 2{,}525p_2M.$$

The value of the factor M is to be estimated from data furnished by the manufacturers of towers as outlined in detail on pages 340 to 341. As there stated, the manufacturer is requested to furnish bids on two series of anchor and suspension towers of the type or types considered. (Only type A tower is being considered in this problem) In the first series on which bids are requested, the towers are all of a given height (45 ft. in this instance) and all are designed for the same tension (6,000 lb), but are to be built for three different voltages, namely, 100, 150 and 200 kv., the range in voltages being so chosen as to embrace the voltage of the contemplated line. The towers of the second series are all to be built for the same height as the first (45 ft.) and for one of the voltages specified in the first series (150 kv., say), but for three different tensions, one of which is also the tension specified in the first series (3,000, 6,000 and 9,000 lb). It is assumed that the suspension towers will be designed for one-half the conductor tension specified for the anchor towers, and that the total ground wire tension per tower for all towers is equal to the conductor tension specified for the anchor tower.

Table 29, together with a sketch of the type of tower as in Fig. 75 and a set of specifications, will furnish the manufacturer with the information required to make up cost estimates.

Specifications for Towers.—In order that the manufacturer of towers may know precisely for what loads, conductor clearances, tensions, etc., the structures are to be designed, it is necessary to define these quantities through the medium of written specifications. Such specifications will vary with the

locality in which the towers are to be used, the type of construction to be adopted, the judgment of the engineer responsible for the design and other factors. The items enumerated below will generally be covered in more or less detail in such specifications.

1. Conductor horizontal and vertical spacings. These will usually be determined upon in some such manner as indicated in the design of the towers illustrated in Fig. 75. The spacings given by Eq. (580) and as shown in Fig. 76 are representative of good practice.
2. Clearances of conductors to nearest tower members.
3. Location of guard cables and clearances to line conductors.
4. Materials to be used.
5. Galvanizing.
6. Foundations.
7. Types of towers, anchor and suspension towers.
8. Loads assumed:
   a. Wind on towers and cables
   b. Maximum tension in cable
   c. Weight of towers, insulators, ice and cables.
9. Unit stresses to be used in calculations:
   a. Tension
   b. Compression
   c. Shear on bolts and rivets
   d. Bearing on bolts and rivets.
10. Total number of anchor towers required.
11. Total number of suspension towers required.

The voltages, tensions, conductor clearances and tower heights, pertaining to the present problem and in accordance with which the manufacturer is requested to furnish bids on both anchor and suspension towers, are given in Table 29. These data together with the detailed specifications covering such items as those already enumerated under specifications, will furnish all the information the manufacturer may need to make up his cost estimates. It is assumed that all such data have been supplied.

The number of towers of each kind required for the present project is estimated by assuming the length of the average span to be 700 ft., and that there is one anchor tower for each seven suspension towers.

Then,

$$T_s + T_a = 7.$$

But,

$$T_s + T_a = 8T_a = \text{total number of towers}$$

or

$$8T_a = \frac{5{,}280 + 200}{700}$$
$$= 1{,}509$$

whence

$$T_a = 189$$

and

$$T_s = 1{,}320.$$

After prices have been secured from the manufacturer for the ten different towers of Table 29, these prices are itemized as in Table 30. In this table items 1, 2 and part of 4 are those which the tower manufacturer furnishes. To the part of item 4, the cost of foundation as furnished, must be added the estimated cost of excavation and back filling to secure the final item in the table. Item 3 is estimated from the total cost of the right-of-way divided by the number of tower locations. Item 6 is estimated from prices of insulator discs and hardware, as furnished by manufacturers' representative or jobbers. By allowing about 17,000 volts per unit of a string, the number of insulators per string (and the total number required) for any assumed line

TABLE 29

Index	Line voltage, assumed, (kilo-volts)	Conductor tension, assumed, pounds	Ground cable tension, assumed, pounds	Spacing D (Fig 76)	Clearance $a = D \div 3.31$ (Eq 577)	Length of insulator string $= 2a - \sqrt{3}$	Tower height, assumed	Class of tower
1	100	6,000	3,000	10′4″	3.13	3′8″	45′0″	Anchor
2	150	6,000	3,000	15′6″	4.68	5′4″	45′0″	
3	200	6,000	3,000	20′8″	6.25	7′2″	45′0″	
4	100	3,000	3,000	10′4″	3.13	3′8″	48′8″	Suspension
5	150	3,000	3,000	15′6″	4.68	5′4″	50′4″	
6	200	3,000	3,000	20′8″	6.25	7′2″	52′2″	
7	150	3,000	1,500	15′6″	4.68	5′4″	45′0″	Anchor
8	150	9,000	4,500	15′6″	4.68	5′4″	45′0″	
9	150	1,500	1,500	15′6″	4.68	5′4″	50′4″	Suspension
10	150	4,500	4,500	15′6″	4.68	5′4″	50′4″	

TABLE 30 —ITEMS OF TOWER COST

Index	Line voltage, kilovolts	Conductor tension, pounds	Item 1 tower at place of erection	Item 2 erection of tower	Item 3 tower site	Item 4 foundation	Item 5 location and inspection	Item 6 insulators at tower sites	Item 7 placing insulators and cable	Class of tower
1	100	6,000	$379	$104	$30	$205	$40	$110 00	$33 00	Anchor
2	150	6,000	386	106	30	205	40	168 00	33 00	
3	200	6,000	408	113	30	205	40	231 00	33 00	
4	100	6,000	251	68	30	116	40	52 50	33 00	Suspension
5	150	6,000	256	70	30	116	40	79 50	33 00	
6	200	6,000	269	75	30	116	40	109 50	33 00	
7	150	3,000	288	79	30	152	40	168 00	33 00	Anchor
8	150	9,000	495	136	30	250	40	168 00	33 00	
9	150	3,000	203	56	30	96	40	79 50	33 00	Suspension
10	150	9,000	308	85	30	132	40	79 50	33 00	

voltage, may readily be estimated. It is here assumed that the unchained strength of the suspension insulators, the strength of a single string, has been increased by the use of high-strength steel so that it will not be necessary to parallel insulator strings in order to sustain the loads due to the use of large cable diameters. A link is used to connect the top insulator of a string to the crossarm. Its length is such that the overall length of the insulator string is $2a \div \sqrt{3}$, (Fig. 75). The cost of this link is estimated as \$0.50 per string.

In Table 31, the estimated prices of item 6 are shown in tabular form.

TABLE 31

	Anchor tower			Suspension tower		
Line voltage	100,000	150,000	200,000	100,000	150,000	200,000
Number of disks required.	6	9	12	6	9	12
Insulators per string	$ 15 00	$ 22 50	$ 30 00	$ 15 00	$ 22.50	$ 30 00
Conductor clamps per string	2 30	3 50	5 00	1 50	2 00	3 00
Arc rings per string	0 50	1 50	3 00	0 50	1 50	3 00
Link per string	0 50	0 50	0 50	0 50	0 50	0 50
Total per string .	$ 18 30	$ 28 00	$ 38 50	$ 17 50	$ 26 50	$ 36 50
Total per type *A* tower	110 00	168 00	231 00	52 70	79 50	109 50

Items 5 and 7, like item 3, are relatively unimportant, and have practically no influence on the choice of the most economical voltage. They are estimated and entered in Table 30 for the sake of completeness.

By Eq (602) a given cost item for the *average* line support is related to the corresponding values of the item as applied to anchor and suspension towers as follows:

Cost item for average support =

$$\frac{\text{cost item for anchor tower} + \frac{T_s}{T_a}\text{(cost item for suspension towers)}}{1 + \frac{T_s}{T_a}}$$

By the use of this equation the items of cost 1, 2, 3, etc., for the average line support, are calculated, as shown in Table 32.

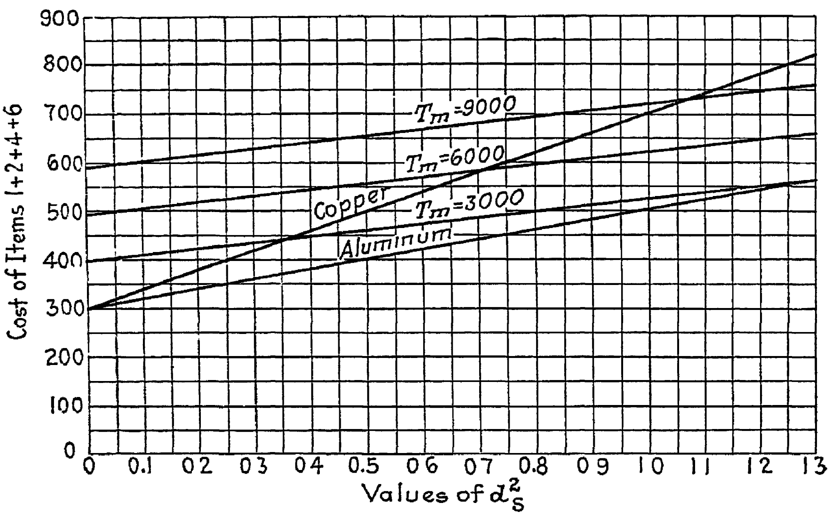

FIG. 101 —Cost items 1 + 2 + 4 + 6 for average line support.

Finding the Values k_v, k_T, m and n.—After the cost items 1, 2, 3, etc., for the average support have been calculated and listed as in Table 32 [items (*a*)], the voltage and tension constants k_v and k_t and the constants m and n, of the same table are computed.

In Fig. 101 are plotted the cost items 1 + 2 + 4 + 6 of the average line support for columns I, II and III [item (*d*)], against the corresponding values of d_s^2 [item (*c*)]. This gives the middle

TABLE 32 —COST OF AVERAGE LINE SUPPORT
Computations for Constants k_v, k_T, m and n

Item	Index	I	II	III	IV	V
	Assumed line voltage (kilovolts)	100	150	200	150	150
	Assumed conductor tension (pounds)	6,000	6,000	6,000	3,000	9,000
(a)	From Table 29 (Eq (602)) — Average line support: Item 1	267 0	272 20	286 40	213 60	331 40
	Combine indices. Item 2	72 50	74 50	79 80	58 90	91 40
	1 and 4 for I — Item 3	30 00	30 00	30 00	30 00	30 00
	2 and 5 for II — Item 4	127 10	127 10	127 10	103 00	146 80
	3 and 6 for III — Item 5	40 00	40 00	40 00	40 00	40 00
	7 and 9 for IV — Item 6	59 70	90 60	124 70	90 60	90 60
	8 and 10 for V — Item 7	33 00	33 00	33 00	33 00	33 00
(b)	Equivalent conductor diameter, $d_s = (E - \sqrt{3}U)$, $U = 116{,}900$	0 494 in	0 741 in	0 988 in	0 741 in	0 741 in
(c)	Equivalent $d_s^2 = E^2 - 3U^2$	0 244	0 549	0 976	0 549	0 549
(d)	Average line support items 1 + 2 + 4 + 6	526 30	561 40	618 00	466 10	660 20
(e)	From Fig 101 and by Eq (603) Cost items $(1 + 2 + 4 + 6) = k_s + md_s^2$	← $k_v = 496$,	$m = 125$	→	$k_v = 397$, $m = 125$	$k_v = 590$, $m = 125$
(f)	d^2 corresponding to T_m (Eq (605)) $d_v^2 = 16\ 1 \times 10^{-5}$		0 966		0 483	1 449
	$d_s^2 = 8\ 01 \times 10^{-5}$		0 482		0 241	0 724
(g)	From Fig 101 k_T for both aluminum and copper		300			
	n for aluminum		200			
	n for copper		403			
(h)	$(m + n)$ for aluminum = 125 + 200		325			
	$(m + n)$ for copper = 125 + 403		528			
(i)	Cost items $(1 + 2 + 4 + 6) = k_T + (m + n)d_v^2$ for aluminum =		$300 + 325d_v^2$			
	for copper =		$300 + 528d_v^2$			

one of the three parallel straight lines of Fig. 101, marked $T_1 =$ 6,000 whose equation is item (*e*), columns I, II, III. The corresponding cost curves for the tensions of 3,000 and 9,000 lb are parallel to the curve already found and pass through the points located by items (*c*) and (*d*) of columns IV and V. The equations of the curve thus located are items (*e*) columns IV and V. Item (*f*), derived from Eq. (605), shows the values of d_s^2 corresponding to the three different tensions and the two kinds of conductor materials. By plotting the three values of k_v (item (*e*)), against the corresponding values of d_s^2 from item (*f*), corresponding to the aluminum and copper conductors, the two curves of Fig 101, which intersect at cost = 300, $d_s^2 = 0$, are obtained. From these curves it is apparent that $k_T = 300$ for both curves, and that $n = 200$ for aluminum and 403 for the copper conductors (item (*g*)). The methods by which items (*h*) and (*i*) are derived are apparent from the table.

The assumed height of the point of suspension of the conductor above ground, in all towers of Table 29, and upon which bids have been obtained, is 45 ft.

By definition (Eq (557), Chap. XII),

$$k_3 = \text{items } (4 + 5 + 6 + 7)$$

and

$$k_2 = \text{items } (1 + 2 + 3)$$

hence

$$\frac{k_3}{k_2} = \frac{h^2 \text{ (items } 4 + 5 + 6 + 7)}{\text{(items } 1 + 2 + 3)}.$$

Thus, if we choose the tower of column II as representing approximately the cost of tower for the present project, its most economical height may be found by the method described in Chap. XII, as follows:

$$\frac{k_3}{k_2} = \frac{(45)^2(127\ 10 + 40.00 + 90.60 + 33.00)}{272.20 \div 74.50 + 30}$$
$$= 1{,}563.$$

By original assumption, the minimum permissible clearance to ground is 30 ft., or

$$k_1 = 30$$

From Plate I, for the above values of $k_3 \div k_2$ and k_1, the conductor sag for the most economical tower spacings or span is found

to be 20.3 ft. Accordingly, the most economical tower height is

$$h_e = 30 + 20.3 = 50\ 3 \text{ ft.}$$

Before the total cost of the towers of height h_e can be found, three items 1, 2, and 3, whose costs vary with the tower height, must be corrected in accordance with Eq. (612). Thus, the correction factor is

$$\begin{aligned} k_4 &= (\text{items } 1 + 2 + 3)\left(\frac{50.3}{45} - 1\right) \\ &= (272.20 + 74.50 + 30)(0\ 25) \\ &= 94.0. \end{aligned}$$

By Eq. (613):

$$\begin{aligned} k_5 &= \text{items } 5 + 7 \\ &= 73\ 0 \\ k &= k_T + k_4 + k_5 \end{aligned}$$

From Eq (615):

$$\begin{aligned} &= 300 + 94 + 73 \\ &= 467. \end{aligned}$$

The total cost of line supports may now be expressed by Eq (617) as soon as the constants k_6, k_7 and k_8 are found These are obtained from Eq (567) and from the charts of Plates 2 to 5, inclusive, by entering the charts with the values $k_3 \div k_2 = 1{,}563$ and $k_1 = 30$. The constants as found from the charts are given below:

$$\left.\begin{aligned} k_6 &= +0\ 20 \\ k_7 &= 978 \\ k_8 &= 428 \end{aligned}\right\} \text{aluminum} \qquad \left.\begin{aligned} k_6 &= -0.15 \\ k_7 &= 840 \\ k_8 &= 114 \end{aligned}\right\} \text{copper.}$$

By substituting in Eq. (619) the values of the above constants, the cost of towers, per foot of line, is expressed in terms of conductor diameter, as shown below.

$$\frac{[(m + n)d_s^2 + k](d_s + k_6)}{k_7(d_s + k_6) - k_8} = \frac{[325d_s^2 + 467](d_s + 0.20)}{978(d_s + 0.20) - 428}, \quad \text{aluminum}$$

or

$$= \frac{[528d_s^2 + 467](d_s - 0.15)}{840(d_s - 0.15) - 114}, \quad \text{copper.}$$

The cost of towers, per foot of line, is found by evaluating the above equations for various assumed values of d_s and plotting the resultant costs as a function of d_s^2. A series of such computations has been made, the results of which are listed in Table 33.

TABLE 33

d_s	k_4d^2		$k_4d_s^2 + k$		$d_s + k_6$		$k_7(d_s + k_6)$	
	Aluminum	Copper	Aluminum	Copper	Aluminum	Copper	Aluminum	Copper
0 5	81 3	132 0	548 3	599 0	0 7	0 35	684 6	294 0
0 6	117 0	190 1	584 0	657 1	0 8	0 45	782 4	378 0
0 7	159 3	258 7	626 3	725 7	0 9	0 55	880 2	462 0
0 8	208 0	337 9	675 0	804 9	1 0	0 65	978 0	546 0
0 9	263 3	427 7	730 3	894 7	1 1	0 75	1,075 8	630 0
1 0	325 0	528 0	792 0	995 0	1 2	0 85	1,173 6	714 0
1 1	393 3	638 9	860 3	1,105 9	1 3	0 95	1,271 4	798 0
1 2	468 0	760 3	935 0	1,227 3	1 4	1 05	1,369 2	882 0
1 3	549 3	892 3	1,016 3	1,359 3	1 5	1 15	1,467 0	966 0
1 4	637 0	1,034 0	1,104 0	1,501 9	1 6	1 25	1,564 8	1,050 0
1 5	731 3	1,188 0	1,198 3	1,655 0	1 7	1 35	1,662 6	1,134 0

TABLE 33 (*Continued*)

d_s	$[k_4d_s^2 + k](d_s + k_6)$		$k_7(d_s + k_6) - k_8$		Cost per foot of line		d_s^2
	Aluminum	Copper	Aluminum	Copper	Aluminum	Copper	
0 5	383 8	209 7	256 6	180 0	$1 50	$1 16	0 25
0 6	467 2	295 7	354 4	264 0	1 32	1 12	0 36
0 7	563 7	399 1	452 2	348 0	1 25	1 15	0 49
0 8	675 0	523 2	550 0	432 0	1 23	1 21	0 64
0 9	803 3	671 0	647 8	516 0	1 24	1 30	0 81
1 0	950 4	845 7	745 6	600 0	1 27	1 41	1 00
1 1	1,118 4	1,050 6	843 4	684 0	1 33	1 54	1 21
1 2	1,309.0	1,288 7	941 2	768 0	1 39	1 68	1 44
1 3	1,524.4	1,563 2	1,039 0	852 0	1 47	1 83	1 69
1 4	1,766 4	1,877 4	1,136 8	936 0	1 55	2 01	1 96
1 5	2,037 1	2,234 2	1,234 6	1,020 0	1 65	2 19	2 25

Plotting the costs of tower per foot of line against values of d_s^2, as shown in Table 33, yields the curves of cost (Fig. 102). In this figure, straight lines are now drawn tangent to the two curves. These tangents are practically coincident with the curves themselves over considerable distances, and, within these ranges, they express the cost of tower per foot of line correctly in terms of the parabolic law, whose equation is Eq. (620), namely,

Cost of towers per foot of line $= Md^2 + N$
From the curves the values of M and N are found to be
For aluminum

$$M = 0\ 275$$
$$N = 1.00$$

For copper

$$M = 0.612$$
$$N = 0.80$$

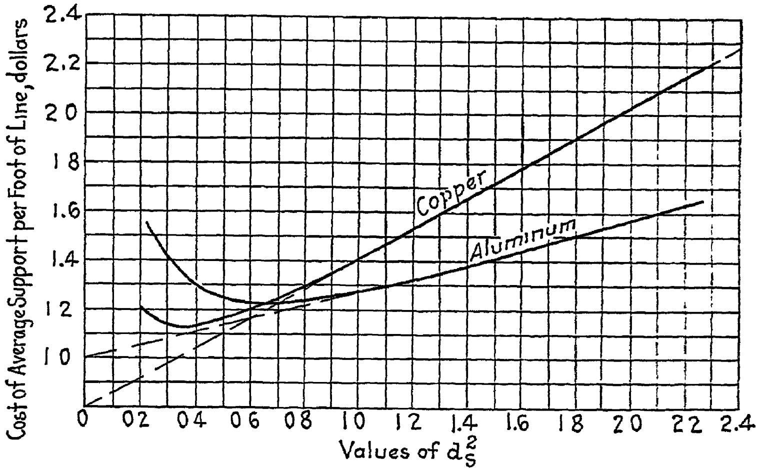

FIG 102 —Cost of average line support as a function of cable diameter

Assuming the depreciation and interest on tower investment at 12 per cent, the item G of Eq. (635) becomes

$$G = 2{,}525 p_2 M$$
$$= 2{,}525 \times 12 \times M$$

whence

$$G_{Al} = 8{,}330, \text{ for aluminum}$$

and

$$G_{Cu} = 18{,}540, \text{ for copper.}$$

Evaluation of Constant *H*, Eq. (635).—The example here considered is for a substation of 100,000-kva. capacity. It is assumed that the substation layout is comparable to type (Fig. 82), as regards the degree of complexity. Experience indicates that in all probability the economical transmission voltage for a 200-mile line having 100,000 kva. of connected generating capac-

ity will exceed 132 kv. The constants are therefore chosen from Table 25 from the columns representing a range in voltage from 132 to 220 kv. The constants are

$$k'_{11} = 11.1 \times 10^{-6}$$
$$k_{12} = 220{,}000$$

By Eq. (628),

$$k_{11} = 3k'_{11}\, U^2.$$

Table 27,

$$U^2 = 136\ 7 \times 10^8.$$

Hence

$$\begin{aligned} k_{11} &= 3 \times 11.1 \times 10^{-6} \times 136.7 \times 10^8 \\ &= 455{,}200. \end{aligned}$$

By Eq. (635), the factor pertaining to terminal equipment is

$$H = \frac{2{,}525 p_3 k_{11}}{L}.$$

The factor p_3, covering the percentage of interest and depreciation on terminal apparatus and housing, is here assumed as 12. The length of line is

$$\begin{aligned} L &= 200 \times 5{,}280 \\ &= 1{,}056 \times 10^3 \text{ ft.} \end{aligned}$$

Solving for the constant H,

$$\begin{aligned} H &= \frac{2.525 \times 12 \times 455{,}200}{1{,}056 \times 10^3} \\ &= 13{,}060 \text{ for both aluminum and copper conductors.} \end{aligned}$$

Solution of Eq. (635). Calculation of E and d_s.—All of the required terms of Eq. (635) have now been evaluated, and the conductor diameter may be found by substitution therein. For convenience, the constants already calculated, together with the solution of Eq. (635) for d_s and the corresponding value of the most economical voltage for the line, are collected below.

The total cost of the line and terminal equipment, for each of the two kinds of conductor materials, may now be estimated, by substituting in the appropriate equations the constants already computed. The results of these calculations and the equations from which the values were obtained are clearly indicated in Table 34. The estimates show that the total costs of the aluminum and copper lines are substantially equal.

TABLE 34

Item	Conductor material	
	Aluminum	Copper
R m.s. kilowatts per line	53,700	53,700
J, loss factor (Table 27)	12.0×10^{10}	19.8×10^{10}
F, line conductors (Table 28)	23,540	36,760
G, line towers	8,330	18,540
H, terminal appliances and housing	13,060	13,060
$J(F + G + H)$	53.92×10^{14}	135.35×10^{14}
$[J(F + G + H)]^{\frac{1}{4}}$.	418.8	488.2
$10 \times$ (r m.s. kilowatts per line)	377.3	377.3
d_s, [Solution of Eq. (635)]	0.901	0.773
$E = \sqrt{3}d_sU$ = line voltage	182,400	156,500

Choice of Type of Conductor.—There are certain objections to the use of an all-aluminum cable, however, and before making a final decision as to the kind of conductor material to use, the comparative cost estimates should be extended to include steel-reinforced aluminum cable and possibly other types such as tubular conductors. From such a complete estimate the engineer may readily select the most suitable type of conductor. If there is no advantage in one kind of conductor over another, the conductor and line voltage which require the minimum total expenditure would naturally be chosen.

In the illustration here used the copper conductor fits the standard voltage, 154 kv., very well. If a steel-reinforced aluminum cable had been included as a part of the estimate, it would probably have been found to call for a voltage close to 220 kv., while the all-aluminum cable falls in between these extremes. Thus the influence of type of conductor upon the most economical voltage of transmission is well illustrated.

Line Specifications and Performance.—For the purpose of illustrating the remaining calculations, the copper line will be chosen. The diameter of the cable will be assumed as computed, namely $d_s = 0.773$ in.

Some of the important required data pertaining to the construction of this line and its performance are listed in Table 35 which follows. The methods used in making the calculations are well illustrated in columns 1 and 2 of the table.

TABLE 35

Item	Equation	Estimated cost	
		Aluminum	Copper
Line conductors [Eq (625) and Table 27]	100 × Eq (625) − p	$ 779,000	$ 971,000
Tower cost (Eqs (621) and (622)) .	LMd_s^2	236,000	386,000
	LN	1,056,000	845,000
Cost of terminal equipment, housing,[1]	$k'_{11}E^2$	388,000	260,000
Wiring, etc (Eqs. (626) and (628))	k_{12}	220,000	220,000
Total cost of terminal equipment and line,	....	2,679,000	2,682,000

[1] It is assumed that the equipment will be purchased for a standard voltage For the aluminum conductor the voltage, 187 kv, will be assumed as the nearest standard For copper the nearest standard is 154 kv

TABLE 36.—LINE SPECIFICATIONS AND PERFORMANCE DATA

Item	Reference, equations, etc	Quantity
Given data		
R m s kilowatts per line	Page 335	53,700
R m s kilowatts per phase ..		17,900
Generator kilovolt-amperes installed	Page 335	100,000
Assumed generator full-load power factor (approximate value)	Page 336	0 90
Kind of conductor material		Copper
Type of towers used (Fig 75)		*A*
d_s = diameter of stranded cable, in inches	Table 33	0 773 in
$E_{av.}$ = approximate average line voltage	Table 33	157,000
$_nE_{av}$ = approximate average e m f to neutral = 156,500 − $\sqrt{3}$	.	90,360
L = length of line in feet	200 × 5,280	10 56 × 10^5
Conductors and guard cables·		
Circular mils metallic section	$10^6d_s^2$ − (1 151)^2	450,000
Number of strands (standard stranding)		37
Circular mils per strand	450,000 − 37	12,162
Diameter of strand in mils	$(12,162)^{\frac{1}{2}}$	110 3
Maximum allowable tension pounds per square inch = 0 75 × 27,500	0 75 × elastic limit	21,000
Maximum conductor tension = 0 9 × combined tension		6,680
Diameter of strands of galvanized steel ground cable (inches)		
D = actual conductor spacing (inches)	Eq (580)	194 in
D in feet and inches	195 − 12	16 ft 2 in
D' = equivalent delta spacing in inches	1 26 × 195 in	244 in
Towers and insulators		
k_1 = minimum clearance to ground (feet)	Page 333	30
k_3 − k_2 ratio .. .	Page 344	1,563
Maximum cable deflection in feet	Page 345	20 3
h_e = the most economical tower height, anchor tower .	Page 345	50 3
k_6		−0 15
k_7	Page 345	840
k_8	.	114
S = tower spacing in feet . ..	Eq (566)	660
Approximate number of towers for line	L − S	1,600
Number of anchor towers	1,600 − 8	200
Number of suspension towers	1,600 − 200	1,400
a = clearance conductor to tower = E_n − 1,538	Fig 75 and Eq (579)	59 in
Length of suspension insulator string	$2a$ ÷ $\sqrt{3}$	68 in
Number of insulator discs per string ...	Page 340	9
Number of discs for anchor towers .	2 × 3 × 9 × 200	10,800
Number of discs for suspension towers	3 × 9 × 1,400	37,800
Total number of discs		48,600

TABLE 36 (*Continued*)

Item	Reference, equations, etc	Quantity
Electrical line constants:		
ρ = resistivity of medium hard drawn copper at 68° F (ohms per mil-foot)	10 37 − 0 97	10 69
r = resistance of line conductor (ohms)	$1.02\rho L \div$ C M	25 59
x = reactance of line conductor at 60 cycles (ohms)	. .	162 57
b = susceptance one conductor to neutral (mho)	. . . . .	10 45 $\times 10^{-4}$
z = impedance one conductor (ohms)	$(r^2 + x^2)^{\frac{1}{2}}$	164 53
$\sqrt{ZY}$ = complex line angle .	$[jb(r + jx)]^{\frac{1}{2}}$	0 41465/85° 31 63′
$= \alpha + j\beta$		0 41465/δ
α = attenuation constant .	0 41465 (cos δ)	0 032347
β = phase constant	0 41465 (sin δ)	0 413384
Z_0 = complex surge impedance (vector ohms) . .	$\left[\frac{z}{b} \angle -90^\circ + 81^\circ\ 3\ 27'\right]^{\frac{1}{2}}$	395 58 − j30 954
Y_0 = complex surge admittance (vector mho)	$1 - Z_0$	(25 125 + j1 9660) $\times 10^{-4}$
$A = \cosh\alpha \cos\beta + j \sinh\alpha \sin\beta$	$a_1 + ja_2$	0 916246 + j0 012998
$a_1 = \cosh\alpha \cos\beta$	1 00052 × 0 91577	0 916338
$a_2 = \sinh\alpha \sin\beta$	0 032357 × 0 40171	0 012998
$B = Z_0(\sinh\alpha \cos\beta + j \cosh\alpha \sin\beta)$	$b_1 + jb_2$	24 1627 + j158 074
b_1 = 395 58($\sinh\alpha \cos\beta$) + j30 954 ($\cosh\alpha \sin\beta$)		24 1627
b_2 = 395 58($\cosh\alpha \sin\beta$) − j30 954 ($\sinh\alpha \cos\beta$)	. . .	158 074
$C = Y_0(\sinh\alpha \cos\beta + j \cosh\alpha \sin\beta)$	$c_1 + jc_2$	(−0 4569 + j101 565)10^{-5}
$c_1 = 10^{-4}$[25 125($\sinh\alpha \cos\beta$) − 1 9660 ($\cosh\alpha \sin\beta$)]	. . . .	−0 4569 $\times 10^{-5}$
$c_2 = 10^{-4}$[25 125($\cosh\alpha \sin\beta$) + 1 9660 ($\sinh\alpha \cos\beta$)]		101 565 $\times 10^{-5}$
A (length of vector)	$(a_1^2 + a_2^2)^{\frac{1}{2}}$	0 916338
B (length of vector)	$(b_1^2 + b_2^2)^{\frac{1}{2}}$	159 910
C (length of vector)	$(c_1^2 + c_2^2)^{\frac{1}{2}}$	101 566 $\times 10^{-5}$
l' = abscissa of power circle center	$\frac{10^3(a_1b_1 + a_2b_2)}{b_1^2 + b_2^2}$	0 94613
m' = ordinate of power circle center	$\frac{10^3(a_1b_2 - a_2b_1)}{b^2 + b_2^2}$	5 6517
n' = radius of power circle for $E_s \div E_r$ = 1.. . .	$10^3 \div (b_1^2 + b_2^2)^{\frac{1}{2}}$	6 2535
Preliminary performance analysis:		
Approximate generator full-load kilowatt output . . .	100,000 × 0 90	90,000
Approximate generator full-load kilowatts per phase. . . .	90,000 ÷ 3	30,000
Approximate generator full-load current (amperes).....	$\frac{100,000,000}{\sqrt{3} \times 160,000}$	360

TABLE 36 (*Continued*)

Item	Reference, equations, etc	Quantity
Estimated full-load line loss (kilowatts = $3rI^2 - 1{,}000$) .	$\frac{3 \times 25\ 3 \times \overline{360}^2}{1{,}000}$	9,800
Approximate receiver full-load kilowatt	90,000 − 9,800	80,200
Hence, assume full-load receiver kilowatts per phase . . .	. .	27,000
Assume receiver power factor at full load		0 90
Assume receiver e m f = approximate 0 95 × 157,000 . .	.	150,000
Then E_r, receiver e m f to neutral, is	$150{,}000 \div \sqrt{3}$	86,700
$10^{-3}Q_r$ = receiver reactance kilovolt-amperes required per phase to maintain any assumed ratio of supply to receiver voltages . . .	$10^{-3}Q_r = 10^{-6}E^2\left[-m' + \sqrt{\frac{n_2'^2E_s^2}{E^2} - \left(\frac{10^3P_r}{E_r^2} + l'\right)^2}\right]$	

Using 86,700 as the receiver voltage to neutral, calculations are made to find the ratio of supply to receiver voltage to neutral, for which the synchronous reactor required is of minimum capacity. If, in the above equation, the reactive power (Q_r) be expressed in kilovolt-amperes, the receiver voltage (E_r) in kilovolts and the receiver power (P_r) in kilowatts, the reactive power required per phase in the receiver circuit is

$$Q_r = E^2\left[-m' + \sqrt{n'^2x^2 - \left(\frac{P_r}{E_r^2} + l'\right)^2}\right]$$

where x is the value of the ratio $E_s \div E_r$, which is now sought. The lagging reactive power required per phase in the receiver circuit at no load is

$$\begin{aligned} Q_{r0} &= E_r^2[-m' + \sqrt{n'^2x^2 - (l')^2}] \\ &= 7{,}516.9[-5.6517 + \sqrt{(6\ 2535)^2x^2 - (0.94613)^2}] \\ &= -42{,}483.2 + 7{,}516.9\sqrt{39.1063x^2 - 0.89516} \\ &= -42{,}483.2 + 47{,}006.1\sqrt{x^2 - 0.022890}. \end{aligned}$$

The reactive power required, per phase in the receiver circuit at full load, is

$$Q_{rL} = E_r^2\left[-m' + \sqrt{n'^2x^2 - \left(\frac{27{,}000}{E_r^2} + l'\right)^2}\right]$$

$$= 7{,}516.9\left[-5.6517 + \sqrt{(6.2535)^2x^2 - \left(\frac{27{,}000}{7516.9} + 0.96413\right)^2}\right]$$

$$= -42{,}483.2 + 7{,}516.9\sqrt{39.1063x^2 - 20.5938}$$
$$= -42{,}483.2 + 47{,}006.1\sqrt{x^2 - 0.52661}.$$

The load power factor has been assumed as constant and equal to 0.90, current lagging. Therefore, at full load, the receiver load supplies lagging reactive kilovolt-amperes to the amount of

$$\begin{aligned} Q_L &= 27{,}000 \times \tan\cos^{-1} 0.90 \\ &= 0.4841 \times 27{,}000 \\ &= 13{,}070 \text{ kva.} \end{aligned}$$

Accordingly, at full load the leading reactive power supplied by the synchronous reactor must be equal to Q_{rL} plus an additional amount sufficient to balance out the lagging reactive kilovolt-amperes of the load, or, at full load,

$$\begin{aligned} Q_{sr} &= Q_{rL} - Q_L \\ &= -13{,}070 - 42{,}483.2 + 47{,}006.1\sqrt{x^2 - 0.52661} \\ &= -55{,}553.2 + 47{,}006.1\sqrt{x^2 - 0.52661}. \end{aligned}$$

If we assume a reactor whose rating as a generator of leading, reactive kilovolt-amperes is equal to 1.5 times its corresponding rating on the lagging side, then it is evident that

$$1\ 5Q_{r0} + Q_{sr} = 0$$

or

$$1.5[-42{,}483.2 + 47{,}006.1\sqrt{x^2 - 0.022890}] - 55{,}553\ 2 + 47{,}006.1\sqrt{x^2 - 0.52661} = 0.$$

This equation, solved for x, yields

$$x = 1.126+$$

the theoretically correct ratio of $E_s \div E_r$.

The supply voltage to neutral is then

$$\begin{aligned} E_s &= 1\ 126 \times 86{,}700 \\ &= 97{,}625 \text{ volts.} \end{aligned}$$

The critical, disruptive voltage for this line at 2,000 ft. elevation is 175,000 volts for equilateral spacing of conductors, or approximately 169,000 volts for the mid-conductor of a flat spaced line. Thus, at the maximum elevation, the mid-conductor will operate approximately at the corona voltage.

Substituting the correct value of x in the appropriate equations for Q yields the values of Q_r for the corresponding assumed loads P_r. These values have been calculated for each 25 per cent of load up to 125 per cent, and are shown in the table on page 355. From this table the lagging kilovolt-amperes per phase furnished by the synchronous reactor is found to be 9,985 kva., while, at full load, the corresponding leading reactive kilovolt-amperes fur-

nished are 1,990 + 13,070 or 15,060. Thus, the rating of the synchronous reactor should be 45,000 kva

The graphical solution for x, corresponding to the foregoing mathematical solution is shown in Fig. 103

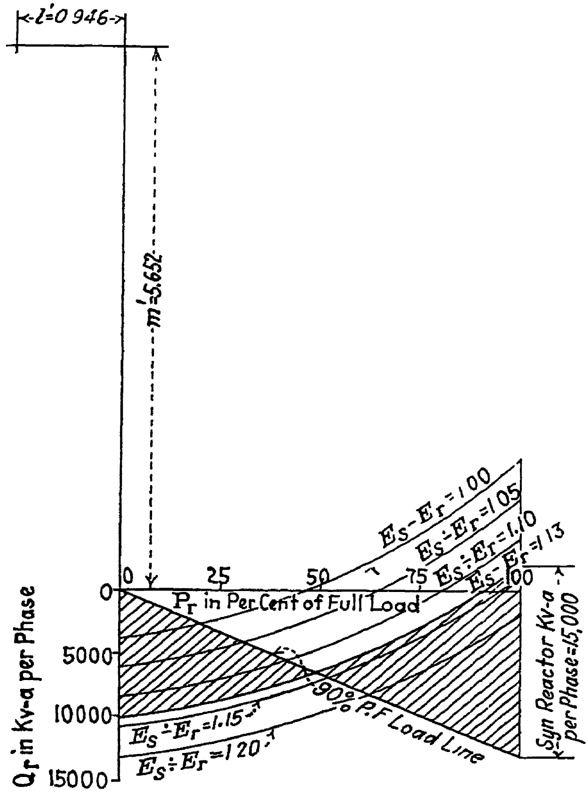

FIG. 103 —Minimum synchronous reactor capacity.

Q_r = NET REACTIVE KILOVOLT-AMPERES IN THE RECEIVER CIRCUIT

	(1)	(2)	(3)[1]	(4)	(5)[2]	(6)[3]
Load, per cent	$\frac{P_r}{E^2}$	$\left(\frac{P_r}{E^2} + l'\right)^2$	49 6180 − (2)	$\sqrt{(3)}$	−5 6517 + (4)	Q_r = 7,516 9 × (5)
0	0 00000	0 89516	48 72286	6 980	1 3283	9,985 (lagging)
25	0 89797	3 40070	46 21732	6 798	1.1466	8,619 (lagging)
50	1 79594	7 51895	42 09907	6 488	0.8366	6,289 (lagging)
75	2 69392	13 24993	36 36809	6 031	0 3793	2,850 (lagging)
100	3 59189	20 59363	29 02439	5 387	−0 2647	−1,990 (leading)
125	4 48986	29 54999	20 06803	4.480	−1.1717	−8,807 (leading)

[1] $n'^2 \times (1.126)^2 = 49.61802$. [2] $-5\ 6517 = -m'$ [3] $7{,}516\ 9 = E_r^2$.

TABLE 37

Item		Reference, equations, etc	Quantity
Data for exact performance diagram			
E_r = receiver voltage to neutral (final value)			86,700
Receiver line voltage		$\sqrt{3} \times 86,700$	150,160
E_s = supply voltage to neutral			97,625
Supply line voltage		$\sqrt{3} \times 97,625$	169,090
$_rI_1$ = full-load active component receiver amperes		$\dfrac{\text{Receiver full load kilowatts}}{3 \times 10^{-3} E_r}$	311 4
$E_1 = E_r\sqrt{a_1^2 + a_2^2}$ components of supply voltage		$86,700 \times 0\ 91634$	79,447
$E_2 = {}_rI_r\sqrt{b_1^2 + b_2^2}$ components of supply voltage		$311\ 4 \times 159\ 991$	49,796
$_sI_2 = E_r\sqrt{c_1^2 + c_2^2}$ components of supply current		$86,700 \times 101\ 566 \times 10^{-5}$	88 1
$_sI_1 = {}_rI_1\sqrt{a_1^2 + a_2^2}$ components of supply current		$311\ 4 \times 0\ 91634$	285 3
Line performance			
Synchronous reactor kilovolt-amperes at full load (total) .		$3 \times 15,000$ kva	45,000
Synchronous reactor kilovolt-amperes at no load (total) .		$3 \times 10,000$ kva	30,000
Total receiver kilowatts for per cent load =	0	$= 3P_r$	0
	25	Values assumed	20,250
	50		40,500
	75		60,750
	100		81,000
	125		101,250
Total receiver reactive kilovolt-amperes for per cent load =	0	$= 3Q_r$	29,960
	25		25,860
	50		18,870
	75		8,550
	100		−5,970
	125		−26,420
Active receiver amperes for per cent load =	0	$_rI_1 = \dfrac{P_r}{E_r}$	0 0
	25		77 8
	50		155 7
	75		233 5
	100		311 4
	125		389 3
Reactive receiver amperes for per cent load =	0	$_rI_2 = \dfrac{Q_r}{E_r}$	115 2
	25		99 4
	50		72 5
	75		32 9
	100		−23 0
	125		−101 6
Supply vector voltage for per cent load =	0	From Eq. (372)	$97,650 - j1,650$
	25		$97,020 + j11,030$
	50		$94,650 + j23,990$
	75		$90,270 + j37,240$
	100		$83,320 + j50,880$
	125		$72,770 + j65,116$

TABLE 37.—(*Continued*)

Item	Per cent load	Reference, equations, etc	Quantity
Angle of supply voltage for per cent load =	0	$\theta_e = \tan^{-1}\frac{E_2}{E_1}$	$-0°\ 53\ 08'$
	25		$6°\ 29\ 3'$
	50		$14°\ 13\ 3'$
	75		$22°\ 25\ 0'$
	100		$31°\ 24\ 6'$
	125		$41°\ 49\ 4'$
Supply vector current for per cent load =	0	From Eq (373)	$1\ 1 - j17\ 5$
	25		$72\ 2 - j2\ 0$
	50		$143\ 2 + j23\ 6$
	75		$214\ 0 + j60\ 9$
	100		$284\ 6 + j113\ 2$
	125		$355\ 1 + j186\ 2$
Angle of supply current for per cent load =	0	$\theta_i = \tan^{-1}\frac{I_2}{I_1}$	$-86°\ 24\ 0'$
	25		$-\ 1°\ 32\ 4'$
	50		$9°\ 25\ 5'$
	75		$15°\ 54\ 5'$
	100		$21°\ 41\ 4'$
	125		$27°\ 40.2'$
Supply current, amperes for per cent load =	0	$\sqrt{I_1^2 + I_2^2}$	17 5
	25		72 2
	50		145 1
	75		222 5
	100		306 3
	125		400 9
Supply power factor angle for per cent load =	0	$\theta = \theta_e - \theta_i$	$85°\ 25\ 9'$
	25		$8°\ 1\ 7'$
	50		$4°\ 47\ 8'$
	75		$6°\ 30\ 5'$
	100		$9°\ 43\ 2'$
	125		$14°\ 9\ 2'$
Supply power factor in per cent for per cent load =	0	$100 \cos \theta$	7 99
	25		99 00
	50		99 65
	75		99 36
	100		98 56
	125		96 96
Supply kilowatts for per cent load =	0	$\frac{3(E_1I_1 + E_2I_2)}{1,000} = 3P_s \div 1,000$	409
	25		20,920
	50		42,360
	75		64,760
	100		88,420
	125		113,900
Line loss, (kilowatts) for per cent load =	0	$3P_L = 3(P_s - P_r)$	409
	25		670
	50		1,860
	75		4,010
	100		7,420
	125		12,650

TABLE 37.—(*Continued*)

Item	Reference, equations, etc	Quantity
Per cent line loss for per cent load = 0	$\left[\dfrac{100 \times 3P_L}{3P_s}\right]$	100 0
25		3 2
50		4 4
75		6 2
100		8 4
125		11 1
Supply voltage to give rated receiver voltage (receiver end open)	$\sqrt{3}E_r\sqrt{a_1^2 + a_2^2}$	137,600
Supply current with rated receiver voltage (receiver end open) . .	$E_r\sqrt{c_1^2 + c_2^2}$	88 1
Receiver voltage with rated supply voltage impressed (receiver end open)	$\dfrac{\sqrt{3}E_s}{\sqrt{a_1^2 + a_2^2}}$	184,550
Per cent voltage rise at receiver (receiver end open and rated supply e m f impressed)	$\dfrac{100(184,550 - 150,160)}{150,160}$	22 9

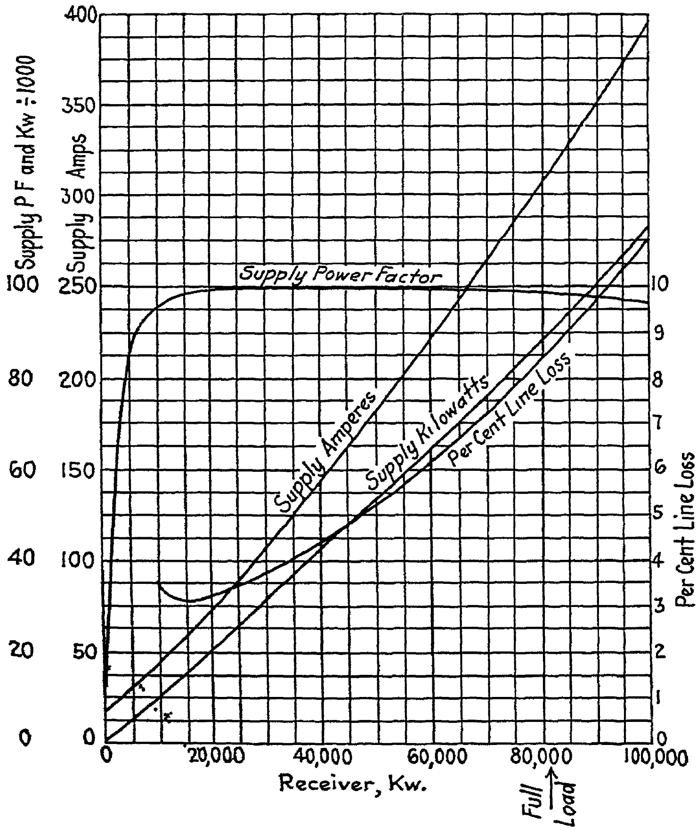

FIG. 105.—Performance curves for 200-mile line

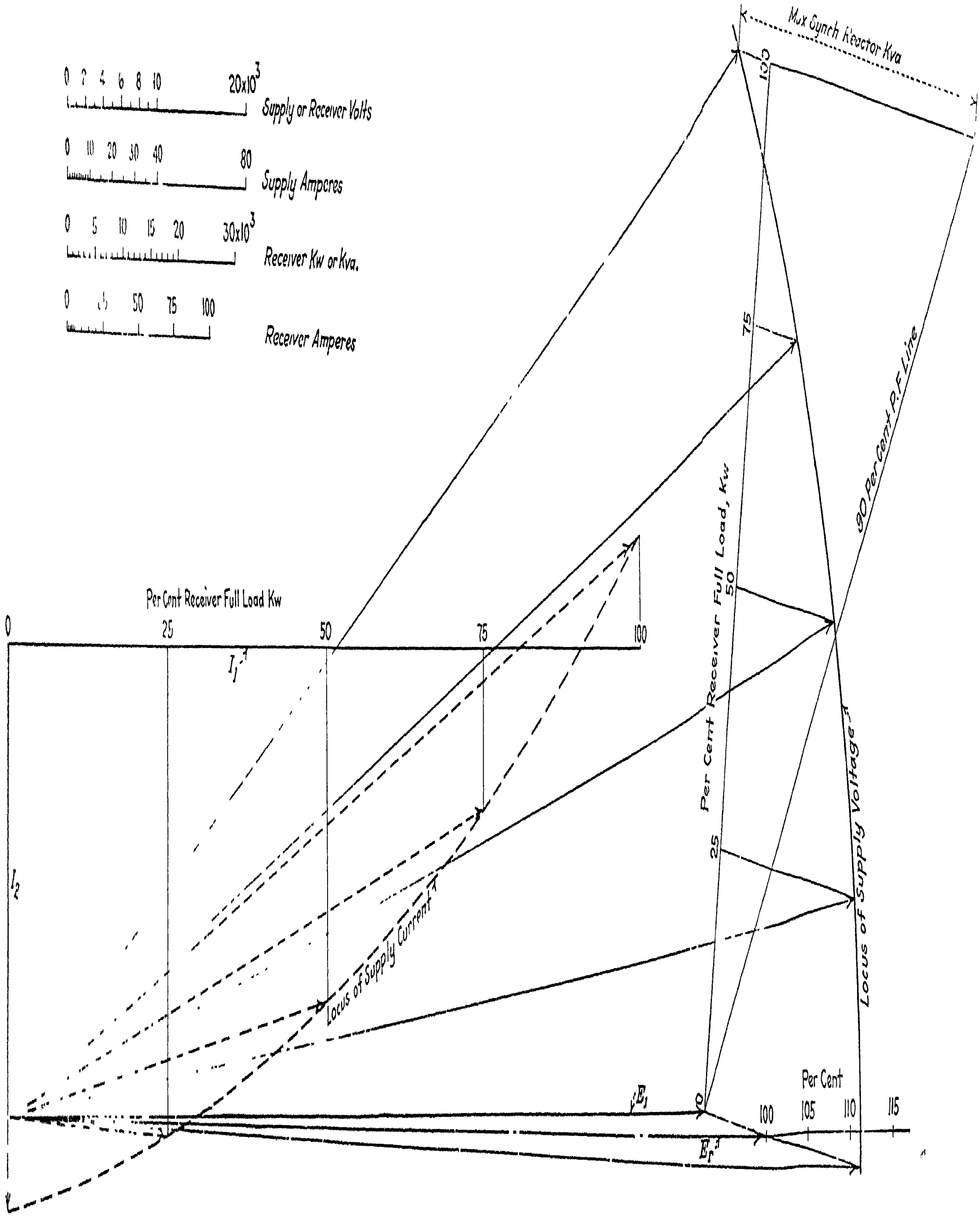

FIG. 104.—Performance diagram for 200-mile line

The performance diagram, representing a graphical solution of the line performance, is shown in Fig 104, while the performance curves, plotted from the above data, are shown in Fig 105.

Temperature-tension stringing charts are not worked out for this line, since the calculations are exactly similar to those of the illustrative problem at Chap. X, Tables 12 to 18, inclusive.

APPENDIX A

Important Relations of Circular and Hyperbolic Trigonometry

$$\operatorname{csch} u = \frac{1}{\sinh u}$$

$$\operatorname{sech} u = \frac{1}{\cosh u}$$

$$\coth u = \frac{1}{\tanh u}$$

$$\frac{d(\sinh u)}{du} = \cosh u$$

$$\frac{d(\cosh u)}{du} = \sinh u$$

$$\frac{d(\tanh u)}{du} = \operatorname{sech}^2 u$$

$$\frac{d(\coth u)}{du} = -\operatorname{csch}^2 u$$

$$\frac{d(\operatorname{sech} u)}{du} = -\operatorname{sech} u \tanh u$$

$$\frac{d(\operatorname{csch} u)}{du} = -\operatorname{csch} u \coth u$$

$$\operatorname{csch}^2 u = \coth^2 u - 1$$

$$\operatorname{sech}^2 u = 1 - \tanh^2 u$$

$$\cosh^2 u - \sinh^2 u = 1$$

$$\sinh (u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v$$

$$\cosh (u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v$$

$$\tanh (u \pm v) = (\tanh u \pm \tanh v) - (1 \pm \tanh u \tanh v)$$

$$\coth (u \pm v) = (\coth u \coth v \pm 1) \div (\coth v \pm \coth u)$$

$$\sinh 2u = 2 \sinh u \cosh u$$

$$\begin{aligned} \cosh 2u &= \cosh^2 u + \sinh^2 u \\ &= 2 \cosh^2 u - 1 \\ &= 1 + 2 \sinh^2 u \end{aligned}$$

$$\tan 2u = 2 \tanh u \div (1 + \tanh^2 u)$$

$$\begin{aligned} \tanh \frac{u}{2} &= (\cosh u - 1) \div \sinh u \\ &= \sinh u \div (1 + \cosh u) \\ &= \sqrt{(\cosh u - 1) \div (\cosh u + 1)} \end{aligned}$$

$$\csc \alpha = \frac{1}{\sin \alpha}$$

$$\sec \alpha = \frac{1}{\cos \alpha}$$

$$\cot \alpha = \frac{1}{\tan \alpha}$$

$$\frac{d(\sin \alpha)}{d\alpha} = \cos \alpha$$

$$\frac{d(\cos \alpha)}{d\alpha} = -\sin \alpha$$

$$\frac{d(\tan \alpha)}{d\alpha} = \sec^2 \alpha$$

$$\frac{d(\cot \alpha)}{d\alpha} = -\csc^2 \alpha$$

$$\frac{d(\sec \alpha)}{d\alpha} = \sec \alpha \tan \alpha$$

$$\frac{d(\csc \alpha)}{d\alpha} = -\csc \alpha \cot \alpha$$

$$\csc^2 \alpha = \cot^2 \alpha + 1$$

$$\sec^2 \alpha = \tan^2 \alpha + 1$$

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos (\alpha \pm \beta) = \cos \alpha \cos \beta \pm \sin \alpha \sin \beta$$

$$\tan (\alpha \pm \beta) = (\tan \alpha \pm \tan \beta) \div (1 \mp \tan \alpha \tan \beta)$$

$$\cot (\alpha \pm \beta) = (\cot \alpha \cot \beta \mp 1) \div (\cot \beta \pm \cot \alpha)$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\begin{aligned} \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cosh^2 \alpha - 1 \\ &= 1 - 2 \sin^2 \alpha \end{aligned}$$

$$\tan 2\alpha = 2 \tan \alpha \div (1 - \tan^2 \alpha)$$

$$\begin{aligned} \tan \frac{\alpha}{2} &= (1 - \cos \alpha) \div \sin \alpha \\ &= \sin \alpha \div (1 + \cos \alpha) \\ &= \sqrt{(1 - \cos \alpha) \div (1 + \cos \alpha)} \end{aligned}$$

APPENDIX B

Natural Hyperbolic Functions

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0 000	1 000 000 000 0		0 000 000 000 0	
001	000 000 500 0	0 000 000 500 0	001 000 000 2	0 001 000 000 2
002	000 002 000 0	000 001 500 0	.002 000 001 3	001 000 001 1
003	000 004 500 0	000 002 500 0	003 000 004 5	001 000 003 2
004	000 008 000 0	000 003 500 0	004 000 010 7	001 000 006 2
0 005	1 000 012 500 0	0 000 004 500 0	0 005 000 020 8	0 001 000 010 1
006	000 018 000 1	.000 005 500 1	006 000 036 0	001 000 015 2
007	000 024 500 1	000 006 500 1	007 000 057 2	001 000 021 2
008	000 032 000 2	000 007 500 1	008 000 085 3	001 000 028 1
009	000 040 500 3	.000 008 500 1	009 000 121 5	001 000 036 2
0 010	1 000 050 000 4	0 000 009 500 1	0 010 000 166 7	0 001 000 045 2
011	000 060 500 6	000 010 500 2	011 000 221 8	001 000 055 1
012	000 072 000 9	000 011 500 3	012 000 288 0	001 000 066 2
013	000 084 501 2	000 012 500 3	013 000 366 2	.001 000 078 2
014	000 098 001 6	000 013 500 4	014 000 457 3	001 000 091 1
0 015	1 000 112 502 1	0 000 014 500 5	0 015 000 562 5	0 001 000 105 2
016	000 128 002 7	000 015 500 6	016 000 682 7	001 000 120 2
017	000 144 503 5	000 016 500 8	017 000 818 8	001 000 136 1
018	000 162 004 4	000 017 500 9	018 000 972 0	001 000 153 2
019	000 180 505 4	000 018 501 0	019 001 143 2	001 000 171 2
0 020	1 000 200 006 6	0 000 019 501 2	0 020 001 331 3	0 001 000 188 1
021	000 220 508 1	000 020 501 5	021 001 543 5	001 000 212 2
022	000 242 009 8	000 021 501 7	022 001 774 7	001 000 231 2
023	000 264 511 7	000 022 501 9	023 002 027 8	001 000 253 1
024	.000 288 013 8	000 023 502 1	024 002 304 1	001 000 276 3
0 025	1 000 312 516 3	0 000 024 502 5	0 025 002 604 3	0 001 000 300 2
026	000 338 019 0	000 025 502 7	026 002 929 4	001 000 325 1
027	000 364 522 1	000 026 503 1	027 003 280 6	001 000 351 2
028	000 392 025 6	000 027 503 5	028 003 658 8	001 000 378 2
029	000 420 529 5	000 028 503 9	029 004 065 0	001 000 416 2
0 030	1 000 450 033 8	0 000 029 504 3	0 030 004 500 2	0 001 000 435 2
031	000 480 538 5	000 030 504 7	031 004 965 4	001 000 465 2
032	000 512 043 7	000 031 505 2	032 005 461 6	001 000 496 2
033	000 544 549 4	000 032 505 7	033 005 989 8	001 000 528 2
034	000 578 055 7	000 033 506 3	034 006 551 1	001 000 561 3
0 035	1 000 612 562 5	0 000 034 506 8	0 035 007 146 3	0 001 000 595 2
036	000 648 070 0	000 035 507 5	036 007 776 5	001 000 630 2
037	000 684 578 1	000 036 508 1	.037 008 442 8	001 000 666 3
038	000 722 086 9	004 037 508 8	038 009 146 0	001 000 704 2
039	000 760 596 4	000 038 509 5	039 009 887 3	001 000 741 3
0 040	1 000 800 106 7	0 000 039 510 3	0 040 010 667 5	0 001 000 780 2
041	000 840 617 7	000 040 511 0	041 011 487 8	001 000 820 3
042	000 882 129 7	000 041 512 0	042 012 349 1	.001 000 861 3
043	000 924 642 4	000 042 512 7	043 013 252 4	001 000 903 3
.044	000 968 156 2	000 043 513 8	.044 014 198 7	001 000 946 3
0 045	1 001 012 670 9	0 000 044 514 7	0 045 015 189 0	0.001 000 991 3

Natural Hyperbolic Functions

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0 045	1 001 012 670 9	0 000 044 514 7	0 045 015 189 0	0 001 000 991 3
046	001 058 186 6	000 045 515 7	046 016 224 4	001 001 035 4
047	001 104 703 3	000 046 516 7	047 017 305 7	001 001 081 3
048	001 152 221 2	000 047 517 9	048 018 434 1	001 001 128 4
049	001 200 740 2	000 048 519 0	049 019 610 5	001 001 176 4
0 050	1 001 250 260 4	0 000 049 520 2	0 050 020 835 9	0 001 001 225 4
051	001 300 781 9	000 050 521 5	051 022 111 4	001 001 275 5
052	001 352 304 7	000 051 522 8	052 023 435 8	001 001 324 4
053	001 404 828 8	000 052 524 1	053 024 816 3	001 001 380 5
054	001 458 354 3	000 053 525 5	054 026 247 8	001 001 431 5
0 055	1 001 512 881 3	0 000 054 527 0	0 055 027 733 4	0 001 001 485 6
056	001 568 409 8	000 055 528 5	056 029 273 9	001 001 540 5
057	001 624 939 8	000 056 530 0	057 030 870 5	001 001 596 6
058	001 682 471 5	000 057 531 7	058 032 524 1	001 001 653 6
059	.001 741 004 9	000 058 533 4	059 034 235 8	001 001 711 7
0 060	1 001 800 540 1	0 000 059 535 2	0 060 036 006 5	0 001 001 770 7
061	001 861 076 9	000 060 536 8	061 037 837 2	001 001 830 7
062	001 922 615 8	000 061 538 9	062 039 729 0	001 001 892 8
063	001 985 156 4	000 062 540 6	063 041 682 8	001 001 953 8
064	002 048 699 1	000 063 542 7	064 043 699 6	001 002 016 8
0 065	1 002 113 243 9	0 000 064 544 8	0 065 045 779 5	0 001 002 079 9
066	002 178 790 7	000 065 546 8	066 047 926 4	001 002 146 9
067	002 245 339 7	000 066 549 0	067 050 138 4	001 002 212 0
068	002 312 890 9	000 067 551 2	068 052 417 4	001 002 279 0
069	002 381 444 5	000 068 553 6	069 054 764 5	001 002 347 1
0 070	1 002 451 000 4	0 000 069 555 9	0 070 057 180 7	0 001 002 416 2
071	002 521 558 8	000 070 558 4	071 059 666 8	001 002 486 1
072	002 593 119 9	000 071 561 1	072 062 224 1	001 002 557 3
073	002 665 683 5	000 072 563 6	073 064 853 4	001 002 629 3
074	002 739 249 7	000 073 566 2	074 067 555 8	001 002 702 4
0 075	1 002 813 818 6	0 000 074 568 9	0 075 070 332 3	0 001 002 776 5
076	002 889 390 4	000 075 571 8	076 073 183 8	001 002 851 5
077	002 965 965 0	000 076 574 6	077 076 111 4	001 002 927 6
078	003 043 542 6	000 077 577 6	078 079 116 1	001 003 004 7
079	003 122 123 3	000 078 580 7	079 082 197 8	001 003 081 7
0 080	1 003 201 706 4	0 000 079 583 8	0 080 085 360 6	0 001 003 162 8
081	003 282 294 0	000 080 587 6	081 088 602 6	001 003 242 0
082	003 363 884 3	000 081 590 3	082 091 925 6	001 003 323 0
083	003 446 477 9	000 082 593 6	083 095 330 7	001 003 405 1
084	003 530 074 9	000 083 597 0	.084 098 818 9	001 003 488 2
0 085	1 003 614 675 5	0 000 084 600 6	0 085 102 391 1	0 001 003 572 2
086	003 700 279 8	000 085 604 3	086 106 048 5	001 003 657 4
087	003 786 887 7	000 086 607 9	087 109 792 0	001 003 743 5
088	003 874 499 4	000 087 611 7	088 113 622 6	001 003 830 6
089	003 963 114 9	000 088 615 5	.089 117 541 4	.001 003 918 8
0 090	1 004 052 734 5	0 000 089 619 6	0 090 121 549 2	0 001 004 007 8

NATURAL HYPERBOLIC FUNCTIONS

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0 090	1 004 052 734 5	0 000 089 619 6	0 090 121 549 2	0 001 004 007 8
091	004 143 358 1	000 090 623 6	091 125 647 2	001 004 098 0
092	004 234 984 8	000 091 626 7	092 129 836 3	001 004 189 1
093	004 327 617 8	000 092 633 0	093 134 117 5	001 004 281 2
.094	004 421 254 1	000 093 636 3	.094 138 491 8	001 004 374 3
0.095	1 004 515 894 8	0 000 094 640 7	0 095 142 960 3	0 001 004 468 5
.096	004 611 540 0	000 095 645 2	096 147 523 9	001 004 563 6
097	004 708 189 9	000 096 649 9	.097 152 183 7	001 004 659 8
.098	004 805 844 4	000 097 651 5	098 156 940 7	001 004 757 0
099	004 904 503 8	.000 098 659 4	.099 161 795 7	001 004 855 0
0 100	1 005 004 168 0	0 000 099 664 2	0 100 166 750 0	0 001 004 954 3
101	005 104 837 3	000 100 669 3	.101 171 804 4	001 005 054 4
102	005 206 511 7	000 101 674 4	.102 176 960 0	001 005 155 6
.103	005 309 191 3	000 102 679 6	.103 182 217 8	001 005 257 8
.104	005 412 876 2	000 103 684 9	.104 187 578 7	001 005 360 9
0 105	1 005 517 566 5	0 000 104 690 3	0 105 193 043 9	0 001 005 465 2
106	.005 623 262 3	000 105 695 8	106 198 614 2	001 005 570 3
.107	.005 729 963 7	000 106 701 4	.107 204 290 7	001 005 676 5
108	.005 837 670 9	000 107 707 2	.108 210 074 4	001 005 783 7
.109	.005 946 383 9	000 108 713 0	.109 215 966 3	001 005 891 9
0 110	1 006 056 102 9	0 000 109 719 0	0 110 221 967 5	0 001 006 001 2
111	006 166 827 8	000 110 724 9	.111 228 078 9	001 006 111 4
112	006 278 559 1	000 111 731 3	.112 234 301 5	001 006 222 6
113	006 391 296 5	000 112 737 4	.113 240 636 4	001 006 334 9
114	.006 505 040 4	000 113 743 9	.114 247 084 5	001 006 448 1
0 115	1 006 619 790 7	0 000 114 750 3	0.115 253 646 8	0 001 006 562 3
116	006 735 547 7	000 115 757 0	.116 260 324 4	001 006 677 6
117	006 852 311 4	000 116 763 7	.117 267 118 2	001 006 793 8
118	006 970 082 0	000 117 770 6	118 274 029 3	001 006 911 1
119	007 088 859 5	000 118 777 5	.119 281 058 7	001 007 029 4
0 120	1 007 208 644 1	0 000 119 784 6	0.120 288 207 4	0 001 007 148 7
121	007 329 436 0	000 120 791 9	.121 295 476 3	001 007 268 9
122	007 451 235 1	000 121 799 1	122 302 866 6	001 007 390 3
123	007 574 041 7	000 122 806 6	.123 310 379 1	001 007 512 5
124	007 697 855 9	000 123 814 2	.124 318 015 0	001 007 635 9
0 125	1 007 822 677 8	0 000 124 821 9	0 125 325 775 1	0 001 007 760 1
126	007 948 507 5	000 125 829 7	.126 333 660 7	001 007 885 6
127	008 075 345 2	000 126 837 7	.127 341 672 6	001 008 011 9
128	008 203 190 9	000 127 845 7	.128 349 811 8	001 008 139 2
129	008 332 044 9	.000 128 854 0	.129 358 079 3	.001 008 267 3
0 130	1 008 461 907 1	0 000 129 863 2	0.130 366 476 2	0.001 008 396 9
131	.008 592 777 9	000 130 870 8	.131 375 003 4	.001 008 527 2
132	.008 724 657 1	000 131 879 2	.132 383 662 1	001 008 658 7
133	008 857 545 2	000 132 888 1	.133 392 453 1	.001 008 791 0
134	.008 991 442 1	.000 133 896 9	.134 401 377 5	.001 008 924 4
0 135	1 009 126 348 0	0 000 134 906 9	0 135 410 436 3	0.001 009 058 8

Natural Hyperbolic Functions

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0 135	1 009 126 348 0	0 000 134 906 9	0 135 410 436 3	0 001 009 058 8
136	009 262 263 0	000 135 915 0	136 419 630 5	001 009 194 2
137	009 399 187 3	000 136 924 3	137 428 961 2	001 009 330 7
138	009 537 121 0	000 137 933 7	138 438 429 3	001 009 468 1
139	009 676 064 2	000 138 943 2	139 448 035 8	001 009 606 5
0 140	1 009 816 017 1	0 000 139 952 9	0 140 457 781 7	0 001 009 745 9
141	009 956 979 8	000 140 962 7	141 467 668 1	001 009 886 4
142	010 098 952 5	000 141 972 7	142 477 696 0	001 010 027 9
143	010 241 935 3	000 142 982 8	143 487 866 4	001 010 170 4
144	010 385 928 3	000 143 993 0	144 498 180 2	001 010 313 8
0 145	1 010 530 931 7	0 000 145 003 4	0 145 508 638 4	0 001 010 458 2
146	010 676 945 6	000 146 013 9	146 519 242 4	001 010 604 0
147	010 823 970 2	000 147 024 6	147 529 992 8	001 010 750 4
148	.010 972 005 6	.000 148 035 4	148 540 890 7	001 010 897 9
149	011 121 052 0	000 149 046 4	149 551 937 2	001 011 046 5
0 150	1 011 271 109 6	0 000 150 057 6	0 150 563 133 1	0 001 011 195 9
151	011 422 178 4	000 151 068 8	151 574 479 7	001 011 346 6
152	011 574 258 6	000 152 080 2	152 585 977 8	001 011 498 1
153	001 727 350 4	000 153 091 8	153 597 628 6	001 011 650 8
.154	011 881 453 9	000 154 103 5	154 609 432 9	001 011 804 3
0 155	1 012 036 569 3	0 000 155 115 4	0 155 621 391 8	0 001 011 958 9
156	012 192 696 7	000 156 127 4	156 633 506 4	001 012 114 6
157	012 349 836 3	000 157 139 6	157 645 777 5	001 012 271 1
158	012 507 988 3	000 158 152 0	158 658 206 4	001 012 428 9
159	012 667 152 8	000 159 164 5	159 670 793 8	001 012 587 4
0 160	1 012 827 330 0	0 000 160 177 2	0 160 683 541 0	0 001 012 747 2
161	012 988 519 9	000 161 189 9	161 696 448 8	001 012 907 8
162	013 150 722 9	000 162 203 0	162 709 518 4	001 013 069 6
163	013 313 939 0	000 163 216 1	163 722 750 6	001 013 232 2
164	013 478 168 5	000 164 229 5	164 736 146 6	001 013 396 0
0 165	1 013 643 411 4	0 000 165 242 9	0 165 749 707 2	0 001 013 560 6
166	013 809 667 9	000 166 256 5	166 763 433 8	001 013 726 6
167	.013 976 938 3	000 167 270 4	167 777 327 0	001 013 893 2
168	014 145 222 6	000 168 284 3	168 791 387 9	001 014 060 9
.169	014 314 521 1	000 169 298 5	169 805 617 8	001 014 229 9
0 170	1 014 484 833 9	0 000 170 312 8	0 170 820 017 3	0 001 014 399 5
171	014 656 161 2	000 171 327 3	171 834 587 8	.001 014 570 5
172	014 828 503 2	000 172 342 0	172 849 330 0	001 014 742 2
173	015 001 859 9	000 173 356 7	173 864 245 1	.001 014 915 1
174	015 176 231 7	000 174 371 8	.174 879 334 1	001 015 089 0
0 175	1 015 351 618 7	0 000 175 387 0	0 175 894 597 9	0 001 015 263 8
176	015 528 021 0	000 176 402 3	.176 910 037 7	001 015 439 8
177	015 705 438 8	000 177 417 8	.177 925 654 3	001 015 616 6
178	015 883 872 3	000 178 443 5	178 941 448 9	001 015 794 6
179	016 063 321 8	000 179 449 5	179 957 422 4	001 015 973 5
0 180	1 016 243 787 2	0 000 180 465 4	0 180 973 575 9	0 001 016 153 5

Natural Hyperbolic Functions

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0 180	1 016 243 787 2	0 000 180 465 1	0 180 973 575 9	0 001 016 153 5
181	016 425 269 0	000 181 481 8	181 989 910 3	001 016 334 4
182	016 607 767 1	000 182 498 1	183 006 126 7	001 016 516 4
183	016 791 281 9	000 183 514 8	184 023 126 2	001 016 699 5
184	016 975 813 4	000 184 531 5	185 040 009 6	001 016 883 4
0 185	1 017 161 361 9	0 000 185 548 5	0 186 057 078 2	0 001 017 068 6
186	017 347 927 6	000 186 565 7	187 074 332 7	001 017 254 5
187	017 535 510 7	000 187 583 1	188 091 771 3	001 017 441 6
188	017 724 111 3	000 188 600 6	189 109 404 1	001 017 629 8
189	.017 913 729 6	000 189 618 3	190 127 222 9	001 017 818 8
0 190	1 018 104 365 8	0 000 190 636 2	0 191 145 231 9	0 001 018 009 0
191	018 296 020 1	000 191 654 3	192 163 132 0	001 018 200 1
192	018 488 692 7	000 192 672 6	.193 181 824 2	001 018 392 2
193	018 682 383 8	000 193 691 1	194 200 409 7	001 018 585 5
194	018 877 093 6	.000 194 709 8	195 219 189 3	001 018 779 6
0 195	1 019 072 822 3	0 000 195 728 7	0 196 238 164 2	0 001 018 974 9
196	019 269 570 0	000 196 747 7	197 257 335 3	001 019 171 1
197	.019 467 337 0	000 197 767 0	198 276 703 7	001 019 368 4
198	019 666 123 5	000 198 786 5	199 296 270 3	001 019 566 6
199	019 865 929 6	000 199 806 1	200 316 036 3	001 019 766 0
0 200	1 020 066 755 6	0 000 200 826 0	0 201 336 002 5	0 001 019 966 2
201	020 268 601 7	000 201 846 1	202 356 170 1	001 020 167 6
202	020 471 468 0	000 202 866 3	203 376 540 1	001 020 370 0
203	020 675 354 8	000 203 886 8	204 397 113 4	001 020 573 3
204	.020 880 262 3	000 204 907 5	205 417 891 1	001 020 777 7
0 205	1 021 086 190 7	0 000 205 928 4	0 206 438 874 3	0 001 020 983 2
206	021 293 140 2	000 206 949 5	207 460 063 8	001 021 189 5
207	021 501 110 9	000 207 970 7	208 481 460 9	001 021 397 1
208	021 710 103 2	000 208 992 3	209 503 066 4	.001 021 605 5
209	021 920 117 1	000 210 013 8	210 524 881 4	001 021 815 0
0 210	1 022 131 153 0	0 000 211 035 9	0 211 546 907 0	0 001 022 025 6
211	022 343 211 0	000 212 058 0	212 569 144 1	001 022 237 1
212	022 556 291 3	000 213 080 3	213 591 593 7	001 022 449 6
.213	022 770 394 2	000 214 102 9	.214 614 257 0	001 022 663 3
214	022 985 519 9	.000 215 125 7	215 637 134 9	001 022 877 9
0 215	1 023 201 668 6	0 000 216 148 7	0.216 660 228 4	0 001 023 093 5
216	023 418 840 4	000 217 171 8	217 683 538 6	001 023 310 2
217	023 637 035 7	000 218 195 3	218 707 066 4	001 023 527 8
.218	.023 856 254 6	000 219 218 9	219 730 813 0	001 023 746 6
219	024 076 497 4	.000 220 242 8	.220 754 779 3	001 023 966 3
0 220	1 024 297 764 3	0 000 221 266 9	0 221 778 966 3	0 001 024 187 0
221	.024 520 055 4	000 222 291 1	.222 803 375 1	001 024 408 8
222	024 743 371 1	000 223 315 7	.223 828 006 8	001 024 631 7
223	024 967 711 5	000 224 340 4	224 852 862 2	001 024 855 4
224	.025 193 076 9	000 225 365 4	225 877 942 5	001 025 080 3
0 225	1 025 419 467 2	0 000 226 390 3	0 226 903 248 7	0 001 025 306 2

NATURAL HYPERBOLIC FUNCTIONS

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0 225	1 025 419 467 2	0 000 226 390 3	0 226 903 248 7	0 001 025 306 2
226	025 646 883 2	000 227 416 0	227 928 781 8	001 025 533 1
227	025 875 325 1	000 228 441 9	228 954 542 8	001 025 761 0
228	026 104 792 6	000 229 467 5	229 980 532 8	001 025 990 0
229	026 335 286 3	000 230 493 7	231 006 752 7	001 026 219 9
0 230	1 026 566 806 2	0 000 231 519 9	0 232 033 203 7	0 001 026 451 0
231	026 799 352 7	000 232 546 5	233 059 886 7	001 026 683 0
232	027 032 926 1	000 233 573 4	234 086 802 8	001 026 916 1
233	027 267 526 4	000 234 600 3	235 113 952 9	001 027 150 1
234	027 503 153 7	000 235 627 3	236 141 338 2	001 027 385 3
0 235	1 027 739 809 2	0 000 236 655 5	0 237 168 959 5	0 001 027 621 3
236	027 977 492 1	000 237 682 9	238 196 818 1	001 027 858 6
237	028 216 202 9	000 238 710 8	239 224 914 9	001 028 096 8
238	028 455 942 0	000 239 739 1	240 253 250 9	001 028 336 0
239	028 696 709 2	000 240 767 2	241 281 827 1	001 028 576 2
0 240	1 028 938 505 7	0 000 241 796 5	0 242 310 644 6	0 001 028 817 5
241	029 181 330 6	000 242 824 9	243 339 701 5	001 029 059 9
242	029 425 185 2	000 243 854 6	244 369 007 6	001 029 303 1
243	029 670 068 9	000 244 883 7	245 398 555 2	001 029 547 6
244	029 915 982 4	000 245 913 5	246 428 348 1	001 029 792 9
0 245	1 030 162 925 7	0 000 246 943 3	0 247 458 387 5	0 001 030 039 4
246	030 410 899 2	000 247 973 5	248 488 674 3	001 030 286 8
247	030 659 903 2	000 249 004 0	249 519 209 6	001 030 535 3
248	030 909 937 7	000 250 034 5	250 549 994 5	001 030 784 9
249	031 161 003 2	000 251 065 5	251 581 029 8	001 031 035 3
0 250	1 031 413 099 9	0 000 252 096 7	0 252 612 316 8	0 001 031 287 0
251	031 666 227 9	000 253 128 0	253 643 856 4	001 031 539 6
252	031 920 387 7	000 254 159 8	254 675 649 6	001 031 793 2
253	032 175 579 3	000 255 191 6	255 707 697 5	001 032 047 9
254	032 431 803 1	000 256 223 8	256 740 001 1	001 032 303 6
0 255	1 032 689 059 4	0 000 257 256 3	0 257 772 561 5	0 001 032 560 4
256	032 947 348 4	000 258 289 0	258 805 379 6	001 032 818 1
257	033 206 670 3	000 259 321 9	259 838 456 5	001 033 076 9
258	033 467 025 4	000 260 355 1	260 871 793 3	001 033 336 8
259	033 728 413 9	000 261 388 5	261 905 390 9	001 033 597 6
0 260	1 033 990 836 2	0 000 262 422 3	0 262 939 250 4	0 001 033 859 5
261	034 254 292 5	000 263 456 3	263 973 372 9	001 034 122 5
262	034 518 783 1	000 264 490 6	265 007 759 3	001 034 386 4
263	034 784 308 1	000 265 525 0	266 042 410 8	001 034 651 5
264	035 050 867 9	.000 266 559 8	267 077 328 3	001 034 917 5
0.265	1 035 318 462 9	0 000 267 595 0	0 268 112 512 9	0 001 035 184 6
266	035 587 093 1	000 268 630 2	269 147 965 6	001 035 452 7
267	035 856 758 9	000 269 665 8	270 183 687 4	001 035 721 8
268	036 127 460 6	000 270 701 7	271 219 679 4	001 035 992 0
269	.036 399 198 4	000 271 737 8	272 255 942 7	001 036 263 3
0 270	1 036 671 972 6	0 000 272 774 2	0 273 292 478 2	0.001 036 535 5

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0 270	1 036 671 972 6	0 000 272 774 2	0 273 292 478 2	0 001 036 535 5
271	036 945 783 4	000 273 810 8	274 329 287 0	001 036 808 8
272	037 220 631 2	000 274 847 8	275 366 370 1	001 037 083 1
273	037 496 516 3	000 275 885 1	.276 403 728 6	001 037 358 5
274	037 773 438 8	000 276 922 5	277 441 363 5	001 037 634 9
0 275	1 038 051 399 0	0 000 277 960 2	0 278 479 275 8	0 001 037 912 3
276	038 330 397 4	000 278 998 4	.279 517 466 6	001 038 190 8
277	038 610 434 1	000 280 036 7	280 555 936 9	001 038 470 3
278	038 891 509 4	000 281 075 3	281 594 687 8	001 038 750 9
279	039 173 623 5	000 282 114 1	282 633 720 3	001 039 032 5
0 280	1 039 456 776 0	0 000 283 153 4	0 283 673 035 4	0 001 039 315 1

APPENDIX C

$\frac{x}{c}$	$\frac{\operatorname{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\operatorname{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\operatorname{Sinh}\frac{x}{c}}$	$\frac{x}{c}$	$\frac{\operatorname{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\operatorname{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\operatorname{Sinh}\frac{x}{c}}$
0 000	0 000 000 000		0 040	0 000 266 688	50 013 337 9
001	000 000 200	2 000 000 300	041	000 280 190	48 794 158 5
002	000 000 650	1 000 000 675	042	000 294 026	47 633 052 2
003	000 001 500	666 667 666	043	000 308 195	46 525 966 3
004	000 002 675	500 001 331	044	000 322 698	45 469 217 2
0 005	0 000 004 160	400 001 668	045	0 000 337 533	44 459 450 0
006	000 006 000	333 335 332	046	000 352 704	43 493 600 3
007	000 008 174	285 716 618	047	000 368 206	42.568 864 3
008	000 010 662	250 002 667	048	000 384 046	41.682 673 4
009	000 013 500	222 225 222	049	000 400 214	40 832 666 8
0 010	0 000 016 670	200 003 330	0 050	0 000 416 718	40 016 674 2
011	000 020 164	181 821 847 1	051	000 433 557	39 232 694 4
012	000 024 000	166 670 666 7	052	000 450 688	38 478 880 9
013	.000 028 169	153 850 488 3	053	000 468 232	37 753 524 9
014	000 032 664	142 861 811 4	054	000 486 070	37 055 046 5
0 015	0 000 037 500	133 338 313 3	0 055	0 000 504 244	36 381 979 9
0 16	000 042 669	125 005 332 3	056	000 522 748	35 732 963 1
0 17	000 048 164	117 652 725 2	057	000 541 588	35 106 730 5
0 18	000 054 000	111 117 111 5	058	000 560 760	34 502 103 9
0 19	000 060 168	105 269 492 1	059	000 580 268	33 917 984 2
0 020	0 000 066 550	100 006 675 5	0 060	0 000 600 108	33 353 346 5
021	000 073 500	95.245 094 7	061	000 620 282	32 807 232 4
022	000 080 668	90 916 424 3	062	000 640 790	32 278 745 6
023	000 088 165	86 964 188 9	063	000 661 632	31 767 047 0
.024	000 096 004	83 341 334 7	.064	000 682 806	31 271 349 2
025	0 000 104 172	80 008 335 0	0.065	0 000 704 300	30 790 914 3
026	000 112 669	76 931 745 3	066	000 726 158	30 325 047 9
027	000 121 503	74 083 074 7	.067	000 748 334	29 873 098 1
028	000 130 671	71 437 906 1	068	000 770 844	29 434 450 6
029	000 140 172	68 975 185 6	069	000 793 688	29 008 527 4
0 030	0 000 150 007	66 676 668 3	0 070	0 000 816 867	28 594 782 8
031	000 160 174	64 526 464 5	071	000 840 377	28 192 702 5
032	000 170 675	62 510 668 2	072	000 864 224	27 801 800 6
033	000 181 510	60 617 062 6	073	000 888 403	27 421 617 4
034	000 192 679	58 834 865 2	074	000 912 916	27 051 718 4
0 035	0 000 204 180	57 154 526 6	0 075	0 000 937 764	26 691 692 5
036	000 216 014	55 567 558 0	076	000 962 945	26 341 149 6
037	000 228 184	54 066 390 2	077	000 988 460	25.999 720 6
038	000 240 684	52 644 249 0	078	001 014 309	25 667 054 6
039	000 253 521	51 295 055 0	079	001 040 478	25 342 819 3
0 040	0 000 266 688	50 013 377 9	0 080	0 001 067 008	25 026 697 8

$\frac{x}{c}$	$\frac{\operatorname{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\operatorname{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\operatorname{Sinh}\frac{x}{c}}$	$\frac{x}{c}$	$\frac{\operatorname{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\operatorname{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\operatorname{Sinh}\frac{x}{c}}$
0 080	0 001 067 008	25 026 697 8	0 120	0 002 401 728	16 706 772 07
081	001 093 859	24 718 390 3	121	002 441 953	16 569 366 95
082	001 121 044	24 417 610 8	122	002 482 513	16 434 220 00
083	001 148 563	24 124 087 1	123	002 523 407	16 301 276 26
084	001 176 415	23 837 560 0	124	002 564 637	16 170 482 41
0 085	0 001 204 601	23 557 782 7	0 125	0 002 606 201	16 041 786 33
086	001 233 122	23 284 519 4	126	002 648 101	15 915 138 13
087	001 261 977	23 017 546 0	127	002 690 335	15 790 490 14
088	001 291 166	22 756 647 7	128	002 732 905	15 667 794 91
089	001 320 689	22 501 619 9	129	002 775 809	15 547 007 09
0 090	0 001 350 546	22 252 266 8	0 130	0 002 819 048	15 428 082 73
091	001 380 738	22 008 401 3	131	002 862 621	15 310 979 80
092	001 411 264	21 769 844 6	132	002 906 531	15 195 655 79
093	.001 442 123	21 536 425 5	133	002 950 775	15 082 071 24
094	001 473 316	21 307 979 8	134	002 995 354	14 970 186 67
0 095	0 001 504 845	21 084 350 6	0 135	0 003 040 269	14 859 964 94
096	001 536 707	20 865 387 3	136	003 085 518	14 751 369 41
097	001 568 904	20 650 945 8	137	003 131 104	14 644 363 98
098	001 601 436	20 440 887 4	138	003 177 024	14 538 914 16
099	001 634 300	20 235 097 4	139	003 223 279	14 434 986 42
0 100	0 001 667 500	20 033 393 7	0 140	0 003 269 869	14 332 548 69
101	001 701 034	19 835 710 2	141	003 316 795	14 231 568 34
102	001 734 902	19 641 907 9	142	003 364 056	14 132 015 11
103	001 769 105	19 451 875 6	143	003 411 653	14 033 859 13
104	001 803 641	19 265 504 5	144	003 459 585	13 937 071 33
0 105	0 001 838 513	19 082 690 13	0 145	0 003 507 851	13 841 622 98
106	001 873 719	18 903 331 00	146	003 556 455	13 747 487 05
107	001 909 259	18 727 330 36	147	003 605 393	13 654 636 38
108	001 945 133	18 554 595 87	148	003 654 667	13 563 044 94
109	.001 981 342	18 385 036 01	149	003 704 277	13 472 687 43
0 110	0 002 017 886	18 218 565 97	0 150	0 003 754 221	13 383 539 67
111	002 054 765	18 055 101 50	151	003 804 501	13 295 576 78
112	002 091 978	17 894 562 16	152	003 855 117	13 208 775 86
113	.002 129 526	17 736 869 58	153	003 906 069	13 123 114 18
114	002 167 407	17 581 950 51	.154	003 957 356	13 038 569 58
115	0 002 205 624	17 429 731 12	0 155	0 004 008 979	12 955 120 06
116	002 244 176	17.280 141 67	156	004 060 938	12 872 714 61
117	002 283 062	17 133 114 83	157	004 113 232	12 791 423 17
118	002 322 282	16 988 586 23	158	004 165 863	12 711 135 34
119	002 361 838	16 846 492 23	159	.004 218 829	12 631 862 08
0.120	0 002 401 728	16 706 772 07	0 160	0.004 272 131	12.553 583 61

$\frac{x}{c}$	$\frac{\text{Sinh}\,\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\text{Cosh}\,\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\text{Sinn}\,\frac{x}{c}}$	$\frac{x}{c}$	$\frac{\text{Sinh}\,\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\text{Cosh}\,\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\text{Sinh}\,\frac{x}{c}}$
0 160	0 004 272 131	12 553 583 61	0 200	0 006 680 013	10 067 155 40
161	004 325 769	12 476 281 83	201	006 747 115	10 017 744 80
162	004 379 743	12 399 938 70	202	006 814 555	9 968 826 91
163	004 434 053	12 324 536 59	203	006 882 332	9 920 394 49
164	004 488 699	12 250 057 97	204	006 950 447	9 872 440 17
0 165	0 004 543 680	12 176 486 46	0 205	0 007 018 899	9 824 957 23
166	004 598 999	12 103 805 70	206	007 087 688	9 777 938 60
167	004 654 653	12 031 999 00	207	007 156 816	9 731 377 56
168	004 710 642	11 961 051 51	208	007 226 281	9 685 267 54
169	004 766 969	11 890 947 66	209	007 296 083	9 639 602 33
0 170	0 004 823 631	11 821 672 61	0 210	0 007 366 224	9 594 375 13
171	004 880 630	11 753 211 83	211	007 436 701	9 549 580 21
172	004 937 965	11 685 551 17	212	007 507 517	9 505 211 01
173	004 995 636	11 618 676 46	213	007 578 624	9 461 261 78
174	005 053 644	11 552 574 84	214	007 650 163	9 417 726 45
175	0 005 111 989	11 487 232 14	0 215	0 007 721 992	9 374 599 26
176	005 170 669	11 422 636 01	216	007 794 160	9 331 874 78
177	005 229 685	11 358 773 92	217	007 866 665	9 289 547 25
178	005 289 039	11 295 632 88	218	007 939 509	9 247 611 45
179	005 348 728	11 233 201 47	219	008 012 691	9 206 061 63
0 180	0 005 408 755	11 171 467 26	0 220	0 008 086 210	9 164 892 72
181	005 469 118	11 110 419 34	221	008 160 068	9 124 099 61
182	005 529 817	11 050 045 84	222	008 234 264	9 083 677 21
183	005 590 854	10 990 336 01	223	008 308 799	9 043 620 57
184	005 652 226	10 931 279 12	224	008 383 672	9 003 924 62
0 185	0 005 713 936	10 872 864 21	0 225	0 008 458 883	8 964 584 77
186	005 775 982	10 815 081 35	226	008 534 433	8 925 595 93
187	005 838 365	10 757 920 12	227	008 610 321	8 886 953 84
188	005 901 086	10 701 370 56	228	008 686 547	8 848 653 79
189	005 964 142	10 645 423 08	229	008 763 112	8 810 691 27
0 190	0 006 027 536	10 590 068 17	0 230	0 008 840 017	8 773 062 03
191	006 091 267	10 535 296 55	231	008 917 258	8 735 761 44
192	006 155 334	10 481 099 15	232	008 994 840	8 698 785 70
193	006 219 739	10 427 466 79	233	009 072 759	8 662 130 27
.194	006 284 481	10 374 391 01	234	009 151 018	8 625 791 07
0 195	0 006 349 560	10 321 863 31	0 235	0 009 229 615	8 589 764 25
196	006 414 976	10 269 879 80	236	009 308 551	8 554 045 69
197	006 480 729	10 218 418 04	237	009 387 826	8 518 631 68
198	006 546 820	10 167 484 28	238	009 467 441	8 483 518 05
199	006 613 248	10 117 065 88	239	009 547 394	8 448 701 22
0 200	0 006 680 013	10 067 155 40	0 240	0 009 627 685	8 414 177 61

$\frac{x}{c}$	$\frac{\operatorname{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\operatorname{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\operatorname{Sinh}\frac{x}{c}}$	$\frac{x}{c}$	$\frac{\operatorname{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\operatorname{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\operatorname{Sinh}\frac{x}{c}}$
0 240	0.009 627 685	8.414 177 61	0 260	0 011 304 809	7 780 047 680
241	.009 708 317	8 379 943 39	261	011 392 233	7 750 920 996
242	009 789 287	8 345 995 04	262	011 479 997	7 722 019 390
243	009 870 597	8 312 329 01	263	011 568 102	7 693 340 291
244	009 952 246	8 278 941 83	264	011 656 546	7 664 881 176
0 245	0 010 034 234	8 245 830 108	0 265	0 011 745 331	7 636 639 556
246	010 116 562	8 212 990 475	266	011 834 457	7 608 612 977
247	010 199 290	8 180 419 627	267	011 923 922	7 580 799 028
248	010 282 236	8 148 114 315	268	012 013 729	7 553 195 327
249	010 365 581	8 116 071 343	269	012 103 876	7.525 799 531
0 250	0 010 449 267	8 084 287 562	0 270	0 012 194 363	7 498 609 332
251	010 533 292	8 052 759 877	271	012 285 191	7 471 622 456
252	010 617 657	8 021 485 240	272	012 376 360	7 444 836 662
253	010 702 361	7 990 460 652	273	012 467 870	7 418 249 741
254	010 787 406	7 959 683 163	274	012 559 720	7 391 859 517
0 255	0 010 872 790	7 929 149 863	0 275	0 012 651 912	7 365 663 847
256	.010 958 514	7.898 857 896	276	012 744 444	7 339 660 616
257	011 044 578	7.868 804 445	277	012 837 317	7 313 847 742
258	011 130 981	7.838 986 738	278	012 930 531	7 288 223 172
259	011 217 725	7 809 402 045	279	013 021 087	7 262 784 882
0 260	0 011 304 809	7 780 047 680	0 280	0 013 117 983	7 237 530 880

APPENDIX D

Derived Constants—Type *A* Tower Construction

Line Constants a_1

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	0 99469	0 97883	0 95250	0 91605	0 86964	0 81375	0 67545	0 50593
	300,000	0 99468	0 97884	0 95256	0 91612	0 86981	0 81411	0 67712	0 51486
	400,000	0 99471	0 97884	0 95258	0 91619	0 87000	0 81449	0 67769	0 51152
	500,000	0 99470	0 97881	0 95260	0 91622	0 87008	0 81464	0 67821	0 51243
	750,000	0 99470	0 97885	0 95262	0 91625	0 87017	0 81482	0 67872	0 51362
	1,000,000	0 99470	0 97885	0 95260	0 91627	0 87019	0 81488	0 67892	0 51406
	1,250,000	0 99470	0 97884	0 95262	0 91626	0 87019	0 81489	0 67897	0 51418
	1,500,000	0 99470	0 97885	0 95262	0 91627	0 87022	0 81491	0 67904	0 51435
	1,750,000	0 99470	0 97885	0 95262	0 91627	0 87021	0 81492	0 67906	0 51442
	2,000,000	0 99469	0 97885	0 95261	0 91627	0 87022	0 81492	0 67906	0 51444
Copper	250,000	0 99470	0 97883	0 95257	0 91616	0 86990	0 81446	0 67768	0 51126
	300,000	0 99470	0 97883	0 95257	0 91616	0 87000	0 81453	0 67802	0 51210
	400,000	0 99470	0 97884	0 95258	0 91618	0 87012	0 81470	0 67843	0 51324
	500,000	0 99470	0 97883	0 95258	0 91620	0 87018	0 81472	0 67867	0 51351
	750,000	0 99470	0 97881	0 95260	0 91626	0 87019	0 81488	0 67896	0 51420
	1,000,000	0 99470	0 97885	0 95263	0 91628	0 87022	0 81492	0 67904	0 51437
	1,250,000	0 99470	0 97885	0 95263	0 91628	0 87027	0 81495	0 67912	0 51450
	1,500,000	0 99470	0 97886	0 95265	0 91631	0 87029	0 81503	0 67926	0 51475
	1,750,000	0 99470	0 97886	0 95263	0 91630	0 87026	0 81500	0 67920	0 51467
	2,000,000	0 99470	0 97886	0 95262	0 91628	0 87021	0 81497	0 67916	0 51460
Steel	250,000	0 99155	0 96458	0 91948	0 85397	0 76620	0 65381	0 34567	−0 08469
	300,000	0 99128	0 96497	0 92055	0 85806	0 77288	0 66881	0 39021	+0 01476
	400,000	0 99130	0 96514	0 92116	0 85621	0 78083	0 68360	0 43526	0 11725
	500,000	0 99128	0 96555	0 92183	0 86136	0 78405	0 69027	0 45600	0 16402
	750,000	0 99130	0 96560	0 92231	0 86283	0 78781	0 69725	0 47640	0 21179
	1,000,000	0 99131	0 96534	0 92246	0 86333	0 78871	0 69963	0 48394	0 22818
	1,250,000	0 99131	0 96535	0 92250	0 86354	0 78920	0 70182	0 48666	0 23496
	1,500,000	0 99131	0 96535	0 92255	0 86359	0 78940	0 70184	0 48861	0 23904
	1,750,000	0 99131	0 96540	0 92269	0 86381	0 78978	0 70186	0 48971	0 24181
	2,000,000	0 99131	0 96537	0 92264	0 86378	0 78976	0 70193	0 49048	0 24300

Derived Constants—Type *A* Tower Construction

Line Constants a_2

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	0 0024019	0 0095584	0 021313	0 037422	0 57530	0 081213	0 13711	0 20012
	300,000	0 0020064	0 0079831	0 017802	0 031258	0 048056	0 067831	0 11454	0 16651
	400,000	0 0015050	0 0059875	0 013353	0 023446	0 036046	0 050886	0 085929	0 12542
	500,000	0 0012032	0 0047871	0 010678	0 018747	0 028819	0 040684	0 068698	0 10030
	750,000	0 0002025	0 0031929	0 0071201	0 012504	0 019221	0 027135	0 045820	0 06690
	1,000,000	0 0006018	0 0023945	0 0053399	0 0093746	0 014412	0 020349	0 034364	0 05017
	1,250,000	0 0004811	0 0019140	0 0042683	0 0074940	0 011521	0 016263	0 027467	0 04011
	1,500,000	0 0004008	0 0015944	0 0035556	0 0062426	0 0095976	0 013548	0 022888	0 03340
	1,750,000	0 0003435	0 0013668	0 0030483	0 0053513	0 0082272	0 011614	0 019612	0 02864
	2,000,000	0 0003006	0 0011958	0 0026668	0 0046818	0 0051283	0 010161	0 017159	0 02505
Copper	250,000	0 0014647	0 0058212	0 012984	0 022795	0 035045	0 049472	0 083531	0 12195
	300,000	0 0012240	0 0048691	0 010861	0 019068	0 029311	0 041379	0 069871	0 10201
	400,000	0 0009178	0 0036513	0 008143	0 014298	0 021979	0 031027	0 052398	0 07650
	500,000	0 0007341	0 0029210	0 006514	0 011432	0 017583	0 024816	0 041910	0 06119
	750,000	0 0004872	0 0019378	0 004324	0 0075902	0 011669	0 016471	0 027820	0 04062
	1,000,000	0 0003667	0 0014588	0 003254	0 0057110	0 0087798	0 012397	0 020929	0 03057
	1,250,000	0 0002922	0 0011660	0 002600	0 0045635	0 0070177	0 0098951	0 016732	0 02443
	1,500,000	0 0002437	0 0009694	0 0021617	0 0037958	0 0058354	0 0082380	0 013911	0 02031
	1,750,000	0 0002088	0 0008307	0 0018525	0 0032525	0 0050012	0 0070590	0 011921	0 01741
	2,000,000	0 0001820	0 0007258	0 0016183	0 0028417	0 0043702	0 0061667	0 010416	0 01520
Steel	250.000	0 015503	0 061162	0 13632	0 23728	0 36057	0 50136	0 81056	1 10510
	300,000	0 012950	0 051328	0 11381	0 19813	0 30117	0 41904	0 67934	0 93199
	400,000	0 0097067	0 038488	0 085340	0 13825	0 22595	0 31454	0 51120	0 70572
	500,000	0 0077673	0 030800	0 068264	0 11890	0 18087	0 25180	0 40962	0 56713
	750.000	0 0051754	0 020527	0 045526	0 079219	0 12059	0 16794	0 27354	0 37976
	1,000,000	0 0038816	0 015401	0 034157	0 059480	0 090461	0 12603	0 20536	0 28532
	1,250,000	0 0031049	0 012309	0 027290	0 047525	0 072302	0 10068	0 16419	0 22828
	1,500,000	0 0025884	0 010261	0 022749	0 039623	0 060273	0 083942	0 13682	0 19036
	1,750,000	0 0022119	0 008771	0 019447	0 033869	0 051523	0 071768	0 11707	0 16275
	2,000,000	0 0019398	0 007688	0 017046	0 029690	0 045173	0 062919	0 10259	0 14271

DERIVED CONSTANTS—TYPE *A* TOWER CONSTRUCTION

Line Constants b_1

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	18 598	36 804	54 265	70 476	85 688	98 104	117 08	125 36
	300,000	15 535	30 740	45 112	58 874	71 185	81 946	97 857	104 47
	400,000	11 653	23 059	33 974	44 164	53 401	61 479	73 404	78 685
	500,000	9 3166	18 435	27 164	35 310	42 694	49 153	58 691	62 921
	750,000	6 2137	12 295	18 118	23 550	28 474	32 784	39 150	41 981
	1,000,000	4 6598	9 2206	13 686	17 659	21 353	24 576	29 361	31 487
	1,250,000	3 7247	7 3702	10 859	14 115	17 068	19 650	23 468	25 168
	1,500,000	3 1029	6 1398	9 0460	11 759	14 219	16 370	19 554	20 966
	1,750,000	2 6598	5 2631	7 7547	10 080	12 188	14 034	16 758	17 973
	2,000,000	2 3270	4 6024	6 7842	8 8189	10 664	12 277	14 662	15 725
Copper	250,000	11 331	22 416	33 029	42 932	51 911	59 763	71 352	76 478
	300,000	9 4710	18 740	27 614	35 893	43 398	49 962	59 654	63 935
	400,000	7 1045	14 058	20 712	26 926	32 544	37 481	44 755	47 989
	500,000	5 6786	11 231	16 546	21 505	26 007	30 012	35 707	38 337
	750,000	3 7706	7 4597	10 994	14 290	17 279	19 891	23 758	25 477
	1,000,000	2 8372	6 0053	8 2710	10 751	12 999	14 968	17 872	19 170
	1,250,000	2 2657	4 4899	6 6156	8 6178	10 398	11 971	14 298	15 335
	1,500,000	1 8909	3 7417	5 5130	7 1671	8 6665	9 9786	11 919	12 788
	1,750,000	1 6163	3 1983	4 7122	6 1255	7 4074	8 5278	10 185	10 924
	2,000,000	1 4079	2 7887	4 1084	5 3407	6 4588	7 4342	8 8785	9 5191
Steel	250,000	119 39	235 04	343 02	437 73	515 62	572 53	607 10	571 75
	300,000	100 15	196 76	286 44	365 89	431 06	479 84	514 16	447 46
	400,000	75 081	147 54	214 81	264 89	323 89	361 11	391 14	351 17
	500,000	60 056	118 04	171 80	219 53	259 26	289 29	314 65	286 43
	750,000	40 028	78 685	114 57	146 35	173 01	193 21	211 09	194 95
	1,000,000	30 033	59 041	85 979	109 88	129 35	145 09	158 81	147 30
	1,250,000	24 011	47 193	68 702	87 801	103 77	116 02	126 99	118 00
	1,500,000	20 011	39 322	57 254	73 177	86 481	96 639	105 88	98 487
	1,750,000	17 107	33 620	48 956	62 569	73 952	82 653	90 588	84 319
	2,000,000	15 725	30 617	43 880	54 748	62 538	66 640	61 966	38 278

DERIVED CONSTANTS—TYPE *A* TOWER CONSTRUCTION

Line Constants b_2

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	41 110	81 870	121 93	160 97	198 64	234 61	300 19	355 30
	300,000	41 105	81 829	121 81	160 68	198 09	233 68	298 09	349 87
	400,000	41 098	81 797	121 69	160 41	197 54	232 76	295 99	347 51
	500,000	41 095	81 776	121 64	160 27	197 29	232 32	295 01	345 79
	750,000	41 094	81 760	121 63	160 15	197 05	231 90	294 06	344 02
	1,000,000	41 093	81 754	121 56	160 09	196 95	231 74	293 71	343 39
	1,250,000	41 094	81 756	121 56	160 08	196 92	231 69	293 56	343 12
	1,500,000	41 095	81 754	121 55	160 06	196 89	231 65	293 53	342 96
	1,750,000	41 093	81 752	121 55	160 06	196 88	231 62	293 43	342 86
	2,000,000	41 094	81 754	121 55	160 06	196 87	231 61	293 39	342 80
Copper	250,000	41 104	81 806	121 71	160 41	197 54	233 29	295 89	347 36
	300,000	41 110	81 785	121 65	160 29	197 33	232 36	295 08	345 78
	400,000	41 094	81 765	121 60	160 17	197 10	232 00	294 28	343 33
	500,000	41 083	81 676	121 46	160 07	196 93	231 61	293 62	343 41
	750,000	41 086	81 750	121.54	160 05	196 91	231 65	293 52	343 05
	1,000,000	41 098	81 741	121 56	160 06	196 89	231 65	293 45	342 90
	1,250,000	41 081	81 724	121 51	160 00	196 81	231 53	293 39	342 78
	1 500,000	41 078	81 715	121 49	160 00	196 77	231 49	293 24	342 61
	1,750 000	41 094	81 731	121 55	160 00	196 81	231 53	293 27	342 63
	2,000,000	41 077	81 722	121 50	160 00	196 79	231 49	293 21	342 56
Steel	250,000	67 870	138 67	213 73	295 94	386 56	486 50	713 31	963 92
	300,000	67 723	136 79	208 57	284 11	364 44	449 26	632 30	823 59
	400 000	67 534	135 37	203 68	275 11	342 17	412 04	550 51	680 98
	500,000	67 487	134 74	201 48	267 38	332 02	394 92	512 68	614 41
	750,000	67 344	133 97	199 10	261 90	321 64	377 66	475 17	548 28
	1,000,000	67 318	133 75	198 30	260 04	315 88	371 70	461 96	525 22
	1,250,000	67 350	133 62	197 87	259 15	316 47	368 89	456 05	514 60
	1,500,000	67 353	133 63	197 78	258 81	315 70	367 54	452 75	508 90
	1,750,000	67 327	133 61	197 52	258 37	314 99	366 43	450 70	505 22
	2,000,000	67 418	133 65	197 61	258 26	314 51	365 38	447 68	499 50

DERIVED CONSTANTS—TYPE A TOWER CONSTRUCTION

Line Constants c_1 = Values in table $\times 10^{-5}$

	Circular mils	Length in miles 50	100	150	200	250	300	400	500
Aluminum	250,000	−0 0217	−0 1640	−0 529	−1 306	− 2 518	− 4 298	− 9 886	− 18 565
	300 000	−0 0178	−0 1378	−0 462	−1 086	− 2 203	− 3 591	− 8 238	− 15 064
	400,000	−0 0129	−0 1043	−0 347	−0 816	− 1 578	− 2 694	− 6 194	− 11 616
	500 000	−0 0104	−0 0828	−0 276	−0 651	− 1 262	− 2 152	− 4 954	− 9 302
	750,000	−0 0070	−0 0551	−0 1875	−0 4336	− 0 8410	− 1 4357	− 3 304	− 6 2034
	1,000,000	−0 0053	−0 0414	−0 1388	−0 3267	− 0 6319	− 1 0766	− 2 4768	− 4 6529
	1,250,000	−0 0043	−0 0329	−0 1110	−0 2608	− 0 5406	− 0 8619	− 1 9799	− 3 7192
	1,500,000	−0 0035	−0 0275	−0 0923	−0 2174	− 0 4202	− 0 7175	− 1 6497	− 3 0990
	1,750,000	−0 0029	−0 0236	−0 0790	−0 1861	− 0 3600	− 0 6152	− 1 4144	− 2 6563
	2,000,000	−0 0026	−0 0221	−0 0692	−0 1631	− 0 3150	− 0 5381	− 1 2375	− 2 3241
Copper	250,000	−0 0113	−0 101	−0 336	−0 792	− 1 531	− 2 620	− 6 02	− 11 30
	300,000	−0 0104	−0 084	−0 279	−0 661	− 1 283	− 2 188	− 5 01	− 9 45
	400,000	−0 0079	−0 063	−0 212	−0 496	− 0 963	− 1 643	− 3 78	− 7 09
	500,000	−0 0059	−0 050	−0 169	−0 400	− 0 771	− 1 320	− 3 05	− 5 68
	750,000	−0 0042	−0 035	−0 111	−0 264	− 0 512	− 0 873	− 2 00	− 3 77
	1,000,000	−0 0032	−0 022	−0 0847	−0 209	− 0 386	− 0 656	− 1 51	− 2 83
	1,250,000	−0 0032	−0 020	−0 0676	−0 160	− 0 308	− 0 525	− 1 21	− 2 27
	1,500,000	−0 0030	−0 0175	−0 0645	−0 143	− 0 269	− 0 452	− 1 02	− 1 91
	1,750,000	−0 0030	−0 0144	−0 0482	−0 123	− 0 219	− 0 374	− 0 860	− 1 61
	2,000,000	−0 0024	−0 0125	−0 0421	−0 122	− 0 191	− 0 327	− 0 751	− 1 41
Steel	250,000	−0 129	−1 176	−3 565	−8 327	−15 989	−27 078	−60 852	−103 130
	300,000	−0 114	−0 885	−2 955	−6 900	−13 379	−22 590	−50 864	− 92 544
	400,000	−0 0828	−0 665	−2 223	−7 286	−10 005	−16 950	−38 168	− 69 622
	500,000	−0 0655	−0 525	−1 785	−4.165	− 8 003	−13 568	−30 534	− 55 826
	750,000	−0 0429	−0 353	−1 190	−2 792	− 5 323	− 9 048	−23 579	− 41 549
	1,000,000	−0 0331	−0 267	−0 890	−2 081	− 3 840	− 6 784	−15 276	− 27 953
	1,250,000	−0 0289	−0 2077	−0 709	−1 666	− 3 203	− 5 402	−12.246	− 22 386
	1,500,000	−0 0222	−0 1780	−0 595	−1 390	− 2 670	− 4 529	−10 196	− 18 653
	1,750,000	−0 0203	−0 1486	−0 5020	−1 183	− 2 277	− 3 860	− 8 719	− 15 955
	2,000,000	−0 0183	−0 1126	−0 4446	−1 006	− 2 002	− 3 391	− 7 643	− 13 997

DERIVED CONSTANTS—TYPE *A* TOWER CONSTRUCTION

Line Constants c_2 = Values in table $\times 10^{-5}$

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	25 741	51 208	76 135	100 24	123 28	145 01	183 55	214 13
	300,000	25 740	51 207	76 126	100 24	123 28	145 01	183 59	214 35
	400,000	25 738	51 206	76 129	100 24	123 29	145 03	183 64	214 37
	500,000	25 739	51 205	76 131	100 24	123 29	145 03	183 67	214 48
	750,000	25 740	51 205	76 130	100 24	123 30	145 04	183 69	214 55
	1,000,000	25 740	51 205	76 130	100 24	123 30	145 04	183 70	214 57
	1,250,000	25 740	51 206	76 129	100 24	123 30	145 04	183 70	214 58
	1,500,000	25 740	51 206	76 128	100 24	123 30	145 04	183 70	214 58
	1,750,000	25 740	51 206	76 130	100 24	123 30	145 04	183 70	214 58
	2,000,000	25 741	51 205	76 130	100 25	123 30	145 05	183 71	214 60
Copper	250,000	25 740	51 226	76 129	100 24	123 29	145 03	183 65	214 42
	300,000	25 742	51 227	76 130	100 29	123 35	145 09	183 73	214 48
	400,000	25 740	51 226	76 130	100 24	123 39	145 03	183 68	214 52
	500,000	25 745	51 224	76 130	100 29	123 38	145 05	183 74	214 64
	750,000	25 747	51 226	76 130	100 28	123 35	145 08	183 75	214 65
	1,000,000	25 745	51 222	76 130	100 26	123 32	145 06	183 73	214 61
	1,250,000	25 742	51 220	76 130	100 25	123 32	145.06	183 72	214 60
	1,500,000	25 736	51 224	76 130	100 24	123 29	145 04	183 70	214 60
	1,750,000	25 738	51 220	76 130	100 24	123 29	145 04	183 69	214 60
	2,000,000	25 739	51 223	76 130	100 28	123 30	145 09	183 70	214 67
Steel	250,000	25 704	50 889	75 300	98 203	119 13	137 50	163 77	171 10
	300,000	25 703	50 939	75 302	98 244	119 40	137 91	165 62	176 47
	400,000	25 700	50 954	75 306	98 350	119 49	138 40	167 55	182 08
	500,000	25 703	50 984	75 345	98 379	119 64	138 68	168 47	184 70
	750,000	25 702	50 972	75 353	98 389	119 70	138 88	169 35	186 24
	1,000,000	25 702	50 974	75 355	98 407	119 70	138 96	169 63	187 13
	1,250 000	25 710	50 974	75 352	98 414	119 72	138 94	169 78	188 52
	1,500,000	25 717	50 984	75 361	98 429	119 77	139 02	169 87	188 78
	1,750,000	25 715	50 962	75 329	98 428	119 77	139 00	169 95	188 94
	2,000,000	25 719	50 973	75 340	98 416	119 77	139 02	169 95	189 02

Derived Constants—Type *B* Tower Construction

Line Constants a_1

	Circular mils	Length in miles: 50	100	150	200	250	300	400	500
Aluminum	250,000	0 99469	0 97878	0 95245	0 91588	0 86936	0 81331	0 67455	0 50428
	300,000	0 99469	0 97880	0 95248	0 91595	0 86955	0 81372	0 67577	0 50715
	400,000	0 99469	0 97880	0 95249	0 91602	0 86974	0 81410	0 67700	0 51028
	500,000	0 99469	0 97880	0 95251	0 91610	0 86983	0 81429	0 67757	0 51139
	750,000	0 99469	0 97880	0 95252	0 91610	0 86991	0 81447	0 67813	0 51276
	1,000,000	0 99469	0 97881	0 95253	0 91610	0 86995	0 81453	0 67832	0 51321
	1,250,000	0 99469	0 97881	0 95253	0 91611	0 86998	0 81457	0 67843	0 51346
	1,500,000	0 99469	0 97881	0 95253	0 91612	0 86998	0 81458	0 67848	0 51353
	1,750,000	0 99469	0 97881	0 95253	0 91612	0 86998	0 81459	0 67850	0 51358
	2,000,000	0 99468	0 97881	0 95253	0 91611	0 86998	0 81461	0 67852	0 51368
Copper	250,000	0 99469	0 97879	0 952471	0 91598	0 86968	0 81401	0 67689	0 50995
	300,000	0 99469	0 97880	0 952508	0 91606	0 86983	0 81427	0 67751	0 51129
	400,000	0 99469	0 97880	0 952511	0 91608	0 86990	0 81443	0 67799	0 51238
	500,000	0 99469	0 97881	0 952512	0 91610	0 86993	0 81448	0 67818	0 51285
	750,000	0 99468	0 97881	0 952518	0 91611	0 86995	0 81454	0 67837	0 51332
	1,000,000	0 99468	0 97881	0 952522	0 91611	0 86996	0 81458	0 67847	0 51353
	1,250,000	0 99468	0 97881	0 952527	0 91612	0 86998	0 81459	0 67851	0 51367
	1,500,000	0 99468	0 97881	0 952530	0 91612	0 86998	0 81460	0 67853	0 51369
	1,750,000	0 99468	0 97881	0 952530	0 91612	0 86998	0 81460	0 67854	0 51370
	2,000,000	0 99468	0 97881	0 952532	0 91613	0 86998	0 81460	0 67855	0 51371
Steel	250,000	0 99106	0 96391	0 91742	0 84986	0 75901	0 64222	0 32014	0 01354
	300,000	0 99109	0 96413	0 91849	0 85323	0 76708	0 66700	0 36924	0 02342
	400,000	0 99109	0 96434	0 91958	0 85670	0 77521	0 67515	0 37740	0 07616
	500,000	0 99110	0 96443	0 92008	0 85818	0 77897	0 68280	0 44172	0 14145
	750,000	0 99110	0 96454	0 92057	0 85972	0 78267	0 69034	0 46436	0 19303
	1,000,000	0 99112	0 96458	0 92074	0 86028	0 78400	0 69300	0 47239	0 21125
	1,250,000	0 99111	0 96459	0 92083	0 86051	0 78458	0 69420	0 47603	0 21961
	1,500,000	0 99112	0 96460	0 92087	0 86066	0 78503	0 69488	0 47805	0 22420
	1,750,000	0 99112	0 96460	0 92091	0 86075	0 78513	0 69529	0 47925	0 22726
	2,000,000	0 99112	0 96461	0 92092	0 86080	0 78523	0 69554	0 47997	0 22866

DERIVED CONSTANTS—TYPE *B* TOWER CONSTRUCTION

Line Constants a_2

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	0 0025273	0 010056	0 022425	0 039374	0 060526	0 085429	0 14422	0 21048
	300,000	0 0021107	0 0083980	0 018730	0 032883	0 050540	0 070866	0 12047	0 17583
	400,000	0 0015835	0 0063001	0 014052	0 024670	0 037925	0 053538	0 090390	0 13458
	500,000	0 0012661	0 0050366	0 011233	0 019724	0 030321	0 042795	0 072269	0 10549
	750,000	0 00084425	0 0033592	0 0074907	0 013152	0 020218	0 028544	0 048197	0 070363
	1,000,000	0 00063347	0 0025205	0 0056208	0 0098663	0 015166	0 021417	0 036159	0 052793
	1,250,000	0 00050608	0 0020136	0 0044891	0 0078815	0 012117	0 017109	0 028891	0 042177
	1,500,000	0 00042156	0 0016772	0 0037406	0 0065672	0 010095	0 014248	0 024070	0 035131
	1,750,000	0 00036112	0 0014368	0 0032040	0 0056250	0 0086477	0 012207	0 020612	0 030098
	2,000,000	0 00031610	0 0012577	0 0028046	0 0049248	0 0075712	0 010686	0 018046	0 026344
Copper	250,000	0 0015412	0 0061318	0 013677	0 0240085	0 036915	0 052107	0 097991	0 12840
	300,000	0 0012844	0 0051094	0 011396	0 0200073	0 030759	0 043419	0 073308	0 106935
	400,000	0 00096269	0 0038297	0 0085400	0 0149963	0 023056	0 032544	0 054951	0 080216
	500,000	0 00077180	0 0030710	0 0068480	0 0120220	0 018486	0 026091	0 044062	0 064324
	750,000	0 00051472	0 0020479	0 0045815	0 0080171	0 012325	0 017397	0 029380	0 042893
	1,000,000	0 00038598	0 0017408	0 0034250	0 0060128	0 009244	0 013048	0 022031	0 032171
	1,250,000	0 00030844	0 0012272	0 0027368	0 0048048	0 007387	0 0104271	0 017606	0 025702
	1,500 000	0 00025701	0 0010226	0 0023903	0 0040035	0 006155	0 0086884	0 014670	0 021418
	1,750,000	0 00021973	0 0008744	0 0019499	0 0034235	0 005263	0 0074296	0 012545	0 018315
	2,000,000	0 00019264	0 0007665	0 0017094	0 0030011	0 004614	0 0065127	0 010997	0 016055
Steel	250,000	0 016317	0 064686	0 14335	0 24940	0 37873	0 52606	0 84784	1 1498
	300,000	0 013626	0 054017	0 11972	0 20832	0 31645	0 43985	0 71097	0 97099
	400,000	0 010217	0 040507	0 089783	0 15625	0 23744	0 33024	0 53538	0 73626
	500,000	0 0081689	0 032341	0 071784	0 12493	0 18988	0 26418	0 42887	0 59165
	750,000	0 0054761	0 021597	0 047873	0 083330	0 12670	0 17629	0 28657	0 39657
	1,000,000	0 0040864	0 016204	0 035915	0 062516	0 095035	0 13228	0 21513	0 29804
	1,250,000	0 0032657	0 012949	0 028701	0 049960	0 075955	0 10572	0 17197	0 23836
	1,500 000	0 0027209	0 010788	0 023914	0 041625	0 063281	0 088082	0 14330	0 19868
	1,750,000	0 0023274	0 0092286	0 020452	0 035606	0 054128	0 075347	0 12257	0 17999
	2,000,000	0 0020402	0 0080867	0 017929	0 031209	0 047442	0 066043	0 10745	0 14902

DERIVED CONSTANTS—TYPE B TOWER CONSTRUCTION
Line Constants b_1

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250 000	18 598	36 800	54 218	70 475	85 201	98 067	117 01	125 23
	300 000	15 534	30 738	45 288	58 866	71 163	81 922	97 767	104 69
	400,000	11 653	23 058	33 974	44 161	53 392	61 464	73 367	79 220
	500 000	9 3185	18 438	27 166	35 310	42 695	49 145	58 674	62 878
	750,000	6 2122	12 293	18 110	23 541	28 462	32 768	39 121	41 936
	1,000 000	4 6600	9 2211	13 585	17 657	21 348	24 580	29 346	31 460
	1,250,000	3 7244	7 3698	10 856	14 112	17 063	19 645	23 456	25 147
	1,500,000	3 1019	6 1378	9 0429	11 755	14 212	16 360	19 537	20 943
	1,750,000	2 6580	5 2595	7 7489	10 072	12 178	14 020	16 740	17 949
	2,000,000	2 3235	4 5975	6 7731	8 8050	10 646	11 255	14 632	15 685
Copper	250,000	11 3389	22 436	33 058	42 9659	51 9521	59 804	71 386	76 4706
	300,000	9 4499	18 6973	27 5491	35 8080	43 2962	49 841	59 497	63 7414
	400,000	7 0847	14 2500	20 6505	26 8448	32 4588	37 5534	44 610	47 8147
	500,000	5 6791	11 2381	16 5591	21 5201	26 0224	29 9546	35 766	38 3386
	750,000	3 7884	7 4951	11 0421	14 3529	17 3539	19 9777	23 854	25 5729
	1,000,000	2 8402	5 6203	8 2802	10 7627	13 0131	14 9908	17 886	19 1881
	1,250,000	2 2697	4 4912	6 6171	8 6010	10 3995	11 9719	14 294	15 3255
	1,500,000	1 8912	3 7422	5 6442	7 1666	8 6652	9 9753	11 911	12 7697
	1,750,000	1 6173	3 2006	4 7152	6 1294	7 4111	8 5318	10 187	10 9218
	2,000,000	1 4179	2 7600	4 1337	5 3732	6 4969	7 4793	8 930	9 5748
Steel	250,000	119 93	235 57	342 60	428 90	513 49	568 70	597 37	615 73
	300,000	100 15	196 73	286 15	364 93	429 62	476 90	506 82	451 79
	400,000	75 098	147 52	214 63	273 86	322 77	386 06	386 07	341 62
	500,000	60 040	117 86	171 61	219 01	258 27	287 65	310 89	271 18
	750,000	40 144	78 659	114 46	146 11	172 42	192 21	208 80	191 62
	1,000,000	30 038	59 015	85 872	109 63	129 38	144 30	157 04	144 13
	1,250,000	24 005	47 158	68 623	88 111	103 40	115 34	125 64	115 57
	1,500,000	19 999	39 289	57 175	73 012	86 162	96 927	104 93	96 479
	1,750,000	17 107	33 608	48 903	62 442	73 697	83 123	89 620	82 646
	2,000,000	14 996	29 456	42 869	55 634	64 600	72 076	78 576	72 473

Derived Constants—Type B Tower Construction
Line Constants b_2

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	39 14	77 96	116 38	153 35	189 30	223 66	286 45	339 45
	300,000	39 14	77 93	116 01	153 06	188 70	222 70	284 27	335 42
	400,000	39.14	77 88	115 87	152 75	188 14	221 70	282 02	331 39
	500,000	39 14	77 87	115 84	152 64	187 90	221 26	281 05	329 48
	750,000	39 13	77 86	115 77	152 49	187 62	220 82	280 01	327 58
	1,000,000	39 13	77 86	115 73	152 44	187 51	220 63	279 62	326 89
	1,250,000	39 13	77 85	115 73	152 42	187 51	220 62	279 53	326 68
	1,500,000	39 13	77 84	115 73	152 40	187 46	220 54	279 39	326 46
	1,750,000	39 13	77 83	115 73	152 39	187 44	220 51	279 33	326 35
	2,000,000	39 12	77 82	115 71	152 36	187 40	220 47	279 26	326 23
Copper	250,000	39 135	77 890	115 91	152 69	188 13	221 67	281 90	331 05
	300,000	39 129	77 863	115 82	152 62	187 88	221 26	281 04	329 51
	400,000	39 128	77 858	115 81	152 52	187 69	220 93	280 26	328 05
	500,000	39 128	77 855	115 80	152 48	187 60	220 77	279 88	327 40
	750 000	39 128	77 850	115 75	152 43	187 50	220 61	279 52	326 67
	1,000 000	39 128	77 845	115 73	152 40	187 50	220 54	279 37	326 41
	1,250,000	39 127	77 834	115 73	152 38	187 43	220 50	279 29	326 33
	1,500,000	39 126	77 837	115 72	152 38	187 42	220 48	279 26	326 28
	1,750,000	39 127	77 839	115 72	152 38	187 42	220 48	279 24	326 18
	2,000,000	39 126	77 837	115 72	152 38	187 41	220 47	279 22	326 15
Steel	250,000	66 060	134 85	208 92	287 36	381 04	481 57	711 52	893 13
	300,000	65 857	133 28	203 68	276 50	357 79	442 67	626 86	810 49
	400,000	65 664	131 69	198 39	265 93	334 35	403 40	541 18	671 80
	500,000	65 569	130 98	195 93	260 23	323 48	385 18	501 41	580 69
	750,000	65 544	130 24	193 38	254 63	312 85	367 28	462 17	533 34
	1,000,000	65 442	129 97	192 66	252 64	309 03	360 97	448 38	509 13
	1,250,000	65 431	129 80	192 29	251 70	307 30	358 07	441 10	497 89
	1 500,000	65 423	129 79	192 07	251 24	306 39	356 48	438 53	491 81
	1,750,000	65 416	129 76	191 94	250 94	305 77	355 52	436 43	488 12
	2,000,000	65 422	129 73	191 86	250 76	305 42	354 91	435 11	485 78

DERIVED CONSTANTS—TYPE *B* TOWER CONSTRUCTION

Line Constants c_1 = Values in table $\times 10^{-5}$

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	−0 0232	−0 1800	−0 6100	−1 4357	− 2 7827	− 4 7557	−10 939	− 20 539
	300,000	−0 0195	−0 1526	−0 5104	−1 2026	− 2 3240	− 3 9750	− 9 1392	− 17 159
	400,000	−0 0142	−0 1150	−0 3813	−0 8974	− 1 7461	− 2 9804	− 6 8556	− 12 497
	500,000	−0 0120	−0 0918	−0 3077	−0 7231	− 1 3967	− 2 3835	− 5 4844	− 10 298
	750,000	−0 0076	−0.0609	−0 2044	−0 4807	− 0 9307	− 1 5885	− 3 6542	− 6 8572
	1,000,000	−0 0052	−0 0447	−0 1519	−0 3600	− 0 6979	− 1 1890	− 2 7405	− 5 1465
	1,250,000	−0 0046	−0 0367	−0 1235	−0 2901	− 0 5599	− 0 9534	− 2 1919	− 4 1153
	1,500,000	−0 0038	−0.0305	−0 1021	−0 2399	− 0 4650	− 0 7947	− 1 8241	− 3 4285
	1,750,000	−0 0035	−0.0265	−0 0883	−0 2070	− 0 3995	− 0 6816	− 1 5659	− 2 9383
	2,000,000	−0 0032	−0.0196	−0 0720	−0 1735	− 0 3403	− 0 5867	− 1 3572	− 2 5571
Copper	250,000	−0 01038	−0 11420	−0 3795	−0 8838	− 1.7050	− 2 9093	−10 3107	− 12 910
	300,000	−0 01179	−0 09345	−0 31089	−0 7324	− 1 4160	− 2 4184	− 5 5628	− 10 460
	400,000	−0 00892	−0 06977	−0 23445	−0 54836	− 1 0604	− 1 8126	− 4 1685	− 7 8283
	500,000	−0 00701	−0 05590	−0 18943	−0 44094	− 0 85072	− 1 4546	− 3 3416	− 6 2765
	750,000	−0 00504	−0 03716	−0 12502	−0 29365	− 0 56810	− 0 97018	− 2 2289	− 4 1859
	1,000,000	−0 00357	−0 02779	−0 09356	−0 22013	− 0 42590	− 0 72725	− 1 6725	− 3 1314
	1,250,000	−0 00273	−0.02238	−0 07490	−0 17585	− 0 32300	− 0 58120	− 1 3363	− 2 5087
	1,500,000	−0 00233	−0 01857	−0 06380	−0 15391	− 0 28343	− 0 48430	− 1 1133	− 2 0905
	1,750,000	−0 00214	−0 01592	−0 05286	−0 12527	− 0 24236	− 0 41383	− 0 9518	− 1.7873
	2,000,000	−0 00183	−0 01394	−0 04674	−0 10983	− 0 21247	− 0 36296	− 0 8344	− 1.5670
Steel	250,000	−0 148	−1 174	−3 932	−7 665	−17 676	−29 914	−67 117	−116 50
	300,000	−0 123	−0 982	−3 285	−6 687	−14 763	−24 994	−56 140	− 97 642
	400,000	−0 0930	−0 736	−2 462	−5 761	−11 073	−18 752	−42 169	− 74 50
	500,000	−0 0738	−0 612	−1 966	−4 607	− 8 855	−14 994	−33 746	− 61 60
	750,000	−0 0860	−0 393	−1 313	−3 074	− 5 912	−10 004	−22 519	− 41 104
	1,000,000	−0 0376	−0 294	−0 985	−2 307	− 4 431	− 7 504	−16 895	− 30 851
	1,250,000	−0 0302	−0 2354	−0 788	−2 037	− 3 540	− 5 999	−13 504	− 24 663
	1,500,000	−0 0250	−0 1965	−0 655	−1 535	− 2 953	− 4 998	−11 252	− 20 552
	1,750,000	−0 0210	−0 1671	−0 561	−1 312	− 2 522	− 4 274	− 9 624	− 17 571
	2,000,000	−0 0190	−0 1479	−0 491	−1 151	− 2 213	− 3 747	− 8 438	− 15 413

DERIVED CONSTANTS—TYPE *B* TOWER CONSTRUCTION

Line Constants c_2 = Values in table $\times 10^{-5}$

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	27 08	53 87	80 09	105 46	129 70	152 55	193 07	225 19
	300,000	27 08	53 87	80 09	105 46	129 70	152 55	193 11	225 33
	400,000	27 08	53 87	80 09	105 46	129 70	152 56	193 16	225 60
	500,000	27 08	53 87	80 09	105 46	129 70	152 56	193 20	225 62
	750,000	27 08	53 87	80 09	105 46	129 70	152 56	193 25	225 65
	1,000,000	27 08	53 88	80 10	105 46	129 71	152 57	193 25	225 67
	1,250,000	27 08	53 88	80 10	105 47	129 71	152 58	193 25	225 68
	1,500,000	27.08	53 88	80 10	105 47	129 72	152 59	193 26	225 70
	1,750,000	27 08	53 88	80 10	105 47	129 72	152 60	193 26	225 71
	2,000,000	27 08	53 88	80 11	105 48	129 73	152 61	193 27	225 72
Copper	250,000	27 093	53 897	80 143	105 51	129 77	152 64	193 25	225 57
	300,000	27 087	53 883	80 129	105 48	129 73	152 56	193 22	225 60
	400,000	27 083	53 873	80 106	105 46	129 71	152 58	193 22	225 62
	500,000	27 081	53 879	80 103	105 47	129 72	152 59	193 23	225 66
	750,000	27 082	53 879	80 102	105 47	129 72	152 59	193 24	225 69
	1,000 000	27 083	53 879	80 101	105 47	129 72	152 59	193 25	225 70
	1,250,000	27 084	53 881	80 104	105 47	129 73	152 60	193 26	225 71
	1,500,000	27 084	53 881	80 104	105 47	129 73	152 60	193 26	225 72
	1,750,000	27 080	53 883	80 105	105 47	129 71	152 60	193 26	225 72
	2 000,000	27.080	53 882	80 106	105 49	129 73	152 60	193 26	225 72
Steel	250 000	27 047	53 608	79 172	103 15	125 07	144 14	170 82	191 14
	300,000	27 049	53 613	79 194	103 27	125 30	144 69	173 03	192 75
	400,000	27 049	53 614	79 211	103 35	125 52	145 24	175 26	193 90
	500,000	27 051	53 614	79 222	103 39	125 63	145 50	176 30	194 36
	750,000	27 050	53 617	79 230	103 42	125 76	145 75	177 30	195 32
	1,000,000	27 047	53 616	79 226	103 42	125 76	145 83	177 66	196 35
	1 250,000	27 048	53 617	79 231	103 42	125 78	145 87	177 82	196 83
	1,500,000	27 050	53 620	79 233	103 44	125 81	145 89	177 92	197 10
	1,750,000	27 049	53 620	79 234	103 44	125 80	145 92	177 97	197 27
	2 000,000	27 052	53 617	79 232	103 44	125 80	145 92	178 01	197 36

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